

Dual polar graphs, the quantum algebra $U_q(\mathfrak{sl}_2)$, and Leonard systems of dual q -Krawtchouk type

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Abstract

In this paper we consider how the following three objects are related: (i) the dual polar graphs; (ii) the quantum algebra $U_q(\mathfrak{sl}_2)$; (iii) the Leonard systems of dual q -Krawtchouk type. For convenience we first describe how (ii) and (iii) are related. For a given Leonard system of dual q -Krawtchouk type, we obtain two $U_q(\mathfrak{sl}_2)$ -module structures on its underlying vector space. We now describe how (i) and (iii) are related. Let Γ denote a dual polar graph. Fix a vertex x of Γ and let $T = T(x)$ denote the corresponding subconstituent algebra. By definition T is generated by the adjacency matrix A of Γ and a certain diagonal matrix $A^* = A^*(x)$ called the dual adjacency matrix that corresponds to x . By construction the algebra T is semisimple. We show that for each irreducible T -module W the restrictions of A and A^* to W induce a Leonard system of dual q -Krawtchouk type. We now describe how (i) and (ii) are related. We obtain two $U_q(\mathfrak{sl}_2)$ -module structures on the standard module of Γ . We describe how these two $U_q(\mathfrak{sl}_2)$ -module structures are related. Each of these $U_q(\mathfrak{sl}_2)$ -module structures induces a \mathbb{C} -algebra homomorphism $U_q(\mathfrak{sl}_2) \rightarrow T$. We show that in each case T is generated by the image together with the center of T . Using the combinatorics of Γ we obtain a generating set L, F, R, K of T along with some attractive relations satisfied by these generators.

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1 Introduction

In this paper we investigate a topic that involves algebraic graph theory, quantum groups, and linear algebra. We focus on three objects that turn out to be closely related. The first object is called a *dual polar graph* [2, 4, 6]. Dual polar graphs are distance-regular [4]. The second object is the quantized universal enveloping algebra $U_q(\mathfrak{sl}_2)$. We refer the reader to [16, 17] for background information on $U_q(\mathfrak{sl}_2)$. The third object is called a *Leonard system of dual q -Krawtchouk type*. In order to explain this concept we start with a more basic notion called a *Leonard pair* [26]. Roughly speaking a Leonard pair consists of two diagonalizable linear transformations on a finite-dimensional vector space, each of which acts in an irreducible tridiagonal fashion on an eigenbasis for the other one. A Leonard system is an “oriented” version of a Leonard pair. The Leonard systems are classified up to isomorphism [26, 27]. We will focus on a family of Leonard systems said to have *dual q -Krawtchouk type*.

Our central results are about how the above three objects are related. Shortly we will summarize these results. First we describe the three objects more precisely. We begin with the definition of a Leonard system. For convenience we take the underlying field to be the complex number field \mathbb{C} .

Definition 1.1. [26] Let d denote a nonnegative integer and let V denote a vector space over \mathbb{C} with dimension $d + 1$. By a *Leonard system* on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(v) below.

- (i) Each of A, A^* is a multiplicity-free element in $\text{End}(V)$.
- (ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A .
- (iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^* .

$$(iv) \quad E_i A^* E_j = \begin{cases} 0 & \text{if } |i - j| > 1 \\ \neq 0 & \text{if } |i - j| = 1 \end{cases} \quad (0 \leq i, j \leq d).$$

$$(v) \quad E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i - j| > 1 \\ \neq 0 & \text{if } |i - j| = 1 \end{cases} \quad (0 \leq i, j \leq d).$$

Definition 1.2. Referring to Definition 1.1, for $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*).

Referring to the Leonard system Φ from Definition 1.1, assume Φ has dual q -Krawtchouk type. As we will see, the eigenvalues of A and A^* have the form

$$\theta_i = h + \kappa q^{d-2i} + v q^{2i-d}, \quad \theta_i^* = h^* + \kappa^* q^{d-2i} \quad (0 \leq i \leq d) \quad (1)$$

where $q, h, h^*, \kappa, \kappa^*, v$ are scalars in \mathbb{C} with $q^2 \neq 1$ and κ, κ^*, v nonzero.

We now recall the algebra $U_q(\mathfrak{sl}_2)$. We will use the equitable presentation [15, 29].

Definition 1.3. [15] Let $U_q(\mathfrak{sl}_2)$ denote the \mathbb{C} -algebra with generators $x, y, z^{\pm 1}$ and relations $zz^{-1} = z^{-1}z = 1$,

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

We now show how a Leonard system of dual q -Krawtchouk type is related to $U_q(\mathfrak{sl}_2)$. The following two theorems are our main results along this line.

Theorem 1.4. Fix $\epsilon \in \{1, -1\}$. Let Φ denote a Leonard system on V as in Definition 1.1. Assume Φ has dual q -Krawtchouk type and let $h, h^*, \kappa, \kappa^*, v$ denote the corresponding parameters from (1). Then there exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on V such that on V ,

$$\begin{aligned} A &= h1 + \epsilon \kappa x + \epsilon v y, \\ A^* &= h^* 1 + \epsilon \kappa^* z. \end{aligned}$$

Theorem 1.5. Fix $\epsilon \in \{1, -1\}$. Let Φ denote a Leonard system on V as in Definition 1.1. Assume Φ has dual q -Krawtchouk type and let $h, h^*, \kappa, \kappa^*, v$ denote the corresponding parameters from (1). Then there exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on V such that on V ,

$$\begin{aligned} A &= h1 + \epsilon\kappa y + \epsilon vx, \\ A^* &= h^*1 + \epsilon\kappa^* z. \end{aligned}$$

We now recall the definition of a dual polar graph. Let b denote a prime power. Let \mathbb{F}_b denote the finite field of order b . Let U denote a finite-dimensional vector space over \mathbb{F}_b endowed with one of the following forms: $C_D(b)$, $B_D(b)$, $D_D(b)$, ${}^2D_{D+1}(b)$, ${}^2A_{2D}(q)$, ${}^2A_{2D-1}(q)$, where $b = q^2$ [4, p. 274]. A subspace W of U is called *isotropic* whenever the form vanishes completely on W . By [6, Theorem 6.3.1] each maximal isotropic subspace of U has dimension D . Define a graph Γ as follows. The vertex set X of Γ consists of the maximal isotropic subspaces of U . Vertices y, z in X are adjacent in Γ whenever $\dim(y \cap z) = D - 1$. Let ∂ denote the path-length distance function for Γ . By [4, p. 276] for $y, z \in X$, $\partial(y, z) = D - \dim(y \cap z)$. By [4, p. 274] the graph Γ is distance-regular with diameter D . We call Γ the *dual polar graph* associated with U .

We have a few comments about Γ . By the *standard module* of Γ we mean the vector space $V = \mathbb{C}^X$ of column vectors with rows indexed by X . Let $A \in \text{Mat}_X(\mathbb{C})$ denote the adjacency matrix of Γ . Let $\theta_0 > \theta_1 > \dots > \theta_D$ denote the eigenvalues of A . For $0 \leq i \leq D$ let E_i denote the projection onto the eigenspace of A associated with the eigenvalue θ_i . For the rest of this section fix a vertex $x \in X$. Let $A^* = A^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ whose diagonal is obtained by rotating row x of $|X|E_1$ by 45 degrees. For $0 \leq i \leq D$, by the i^{th} *subconstituent* of Γ we mean the subspace of V spanned by the vertices at distance i from x . The subconstituents are the eigenspaces of A^* ; for $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ (resp. θ_i^*) denote the corresponding projection (resp. eigenvalue). By [4, Theorem 8.4.2, Theorem 9.4.3] the eigenvalues of A and A^* have the form

$$\theta_i = h + \kappa q^{D-2i} + vq^{2i-D}, \quad \theta_i^* = h^* + \kappa^* q^{D-2i} \quad (0 \leq i \leq D)$$

where $h, h^*, \kappa, \kappa^*, v$ are in \mathbb{C} with κ, κ^*, v nonzero. Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by A, A^* . The algebra T is called the *subconstituent algebra* or *Terwilliger algebra* with respect to x [22]. By [8, p. 157] the algebra T is semisimple. By a T -module we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. Let W denote an irreducible T -module. By the *endpoint* of W we mean $\min\{i | 0 \leq i \leq D, E_i^*W \neq 0\}$. By the *dual endpoint* of W we mean $\min\{i | 0 \leq i \leq D, E_iW \neq 0\}$. By the *diameter* of W we mean $|\{i | 0 \leq i \leq D, E_i^*W \neq 0\}| - 1$. We now show how Γ is related to the Leonard systems of dual q -Krawtchouk type. The following is our third main result.

Theorem 1.6. Let W denote an irreducible T -module. Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. Then $(A|_W; \{E_{t+i}|_W\}_{i=0}^d; A^*|_W; \{E_{r+i}^*|_W\}_{i=0}^d)$ is a Leonard system of dual q -Krawtchouk type.

We now summarize how Γ is related to $U_q(\mathfrak{sl}_2)$. Since T is semisimple, V is a direct sum of irreducible T -modules. For each irreducible T -module W in the sum, combining Theorem

1.6 with Theorem 1.4 and Theorem 1.5 we obtain two $U_q(\mathfrak{sl}_2)$ -module structures on W . This gives two $U_q(\mathfrak{sl}_2)$ -module structures on V . In order to describe them in a coherent fashion we introduce some elements Υ, Ψ in the center of T . These elements act on each irreducible T -module W as $q^{r+t+d-D}I$, $q^{r-t}I$ where r, t, d are the endpoint, dual endpoint, and diameter of W , respectively. We now give our fourth and fifth main results.

Theorem 1.7. *There exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on V such that on V ,*

$$\begin{aligned} A &= h1 + \kappa\Upsilon^{-1}\Psi x + v\Upsilon\Psi^{-1}y, \\ A^* &= h^*1 + \kappa^*\Upsilon^{-1}\Psi^{-1}z. \end{aligned}$$

Theorem 1.8. *There exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on V such that on V ,*

$$\begin{aligned} A &= h1 + \kappa\Upsilon^{-1}\Psi y + v\Upsilon\Psi^{-1}x, \\ A^* &= h^*1 + \kappa^*\Upsilon^{-1}\Psi^{-1}z. \end{aligned}$$

Each of the above $U_q(\mathfrak{sl}_2)$ -module structures on V induces a \mathbb{C} -algebra homomorphism $U_q(\mathfrak{sl}_2) \rightarrow T$. We now describe how their images are related to T . The following is our sixth main result.

Theorem 1.9. *For either of our two \mathbb{C} -algebra homomorphisms $U_q(\mathfrak{sl}_2) \rightarrow T$, let U denote the image. Then the algebra T is generated by U together with the elements $\Upsilon^{\pm 1}, \Psi^{\pm 1}$.*

We now describe some relations in T that we find attractive. In order to state these relations, it is convenient to decompose $A = L + F + R$ where L (resp. F) (resp. R) is the lowering matrix (resp. flattening matrix) (resp. raising matrix) of Γ with respect to x . For vertices y, z of Γ the (y, z) -entry of L (resp. F) (resp. R) is 1 whenever y, z are adjacent and $\partial(x, z) - \partial(x, y)$ is 1 (resp. 0) (resp. -1). Define a diagonal matrix $K \in \text{Mat}_X(\mathbb{C})$ with (y, y) -entry $q^{-2\partial(x, y)}$ for $y \in X$. In other words $K = \sum_{i=0}^D q^{-2i} E_i^*$. By construction $A^* \in \text{span}\{I, K\}$. The algebra T is generated by L, F, R, K . We now describe how L, F, R, K are related. The following is our seventh main result.

Theorem 1.10. *The matrices L, F, R, K satisfy*

$$\begin{aligned} KL &= q^2 LK, & KF &= FK, & KR &= q^{-2} RK, \\ LF - q^2 FL &= (q^{2e} - 1)L, & FR - q^2 RF &= (q^{2e} - 1)R, \\ \frac{q^4}{q^2 + 1} RL^2 - LRL + \frac{q^{-2}}{q^2 + 1} L^2 R &= -q^{2e+2D-2} L, \\ \frac{q^4}{q^2 + 1} R^2 L - RLR + \frac{q^{-2}}{q^2 + 1} LR^2 &= -q^{2e+2D-2} R, \end{aligned}$$

where e is given in the table below:

form	$C_D(b)$	$B_D(b)$	$D_D(b)$	${}^2D_{D+1}(b)$	${}^2A_{2D}(q)$	${}^2A_{2D-1}(q)$
e	1	1	0	2	3/2	1/2

We will repeatedly use the relations in Theorem 1.10.

This paper is organized as follows. In Sections 2–7 we recall some background concerning Leonard systems. In Section 8 we introduce the normalized split basis for a Leonard system. In Section 9 we discuss the intersection matrix of a Leonard system. In Section 10 we recall the tridiagonal relations and Askey-Wilson relations of a Leonard system. In Section 11 we recall the Leonard systems of dual q -Krawtchouk type. In Section 12 we recall $U_q(\mathfrak{sl}_2)$ and describe its finite-dimensional irreducible modules. In Section 13 we prove Theorem 1.4 and Theorem 1.5. In Section 14 we discuss the subconstituent algebra T of a distance-regular graph. In Section 15 we recall the notion of a near polygon. After obtaining some basic facts about near polygons, we focus on a particular type of near polygon called a dual polar graph. In Sections 16–19 we discuss some basic facts about a dual polar graph and its irreducible T -modules. In Sections 20, 21 we discuss some central elements Ω, G, G^* of T that come from the Askey-Wilson relations. We describe the entries of the matrices Ω, G, G^* . In Section 22 we introduce three central elements Υ, Ψ, Λ of T which will be used to relate T to $U_q(\mathfrak{sl}_2)$. In Section 23 we prove Theorem 1.6. In Section 24 we prove Theorem 1.7 and Theorem 1.8. In Section 25 we prove Theorem 1.9. In Section 26 we discuss the matrices L, F, R, K and prove Theorem 1.10. In Section 27 we describe Ω, G, G^* in terms of L, F, R, K . In Section 28 we introduce three central elements C_0, C_1, C_2 of T that involve L, F, R, K . We show that C_0, C_1, C_2 generate the center of T . In Section 29 we show how C_0, C_1, C_2 relate to Ω, G, G^* and Υ, Ψ, Λ . In Section 30 we describe the two $U_q(\mathfrak{sl}_2)$ -module structures from Theorem 1.7 and Theorem 1.8 in terms of L, F, R, K .

2 Leonard pairs

We now begin our formal argument. We start by recalling the notion of a Leonard pair. We will use the following terms. A square matrix X is said to be *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume X is tridiagonal. Then X is said to be *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. We now define a Leonard pair. For the rest of this paper \mathbb{K} will denote a field.

Definition 2.1. [26, Definition 1.1] Let V denote a vector space over \mathbb{K} with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair A, A^* where $A : V \rightarrow V$ and $A^* : V \rightarrow V$ are linear transformations that satisfy (i), (ii) below:

- (i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
- (ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

Note 2.2. It is a common notational convention to use A^* to represent the conjugate-transpose of A . We are not using this convention. In a Leonard pair A, A^* the linear transformations A and A^* are arbitrary subject to (i), (ii) above.

3 Leonard systems

When working with a Leonard pair, it is convenient to consider a closely related object called a Leonard system. To prepare for our definition of a Leonard system, we recall a few concepts from linear algebra. Throughout the paper an algebra is meant to be associative and have a 1, and a subalgebra has the same 1 as the parent algebra. Let d denote a nonnegative integer and let $\text{Mat}_{d+1}(\mathbb{K})$ denote the \mathbb{K} -algebra consisting of all $d+1$ by $d+1$ matrices that have entries in \mathbb{K} . We index the rows and columns by $0, 1, \dots, d$. Let V denote a vector space over \mathbb{K} with dimension $d+1$. Let $\{v_i\}_{i=0}^d$ denote a basis for V . For $A \in \text{End}(V)$ and $B \in \text{Mat}_{d+1}(\mathbb{K})$, we say that B represents A with respect to $\{v_i\}_{i=0}^d$ whenever $Av_j = \sum_{i=0}^d B_{ij}v_i$ for $0 \leq j \leq d$. An element $A \in \text{End}(V)$ is said to be *multiplicity-free* whenever it has $d+1$ mutually distinct eigenvalues in \mathbb{K} . Assume A is multiplicity-free. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A . For $0 \leq i \leq d$ let θ_i denote the eigenvalue of A corresponding to V_i . Define $E_i \in \text{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $j \neq i$ ($0 \leq j \leq d$). Here I denotes the identity of $\text{End}(V)$. We call E_i the *primitive idempotent* of A corresponding to V_i (or θ_i). Observe that (i) $AE_i = \theta_i E_i$ ($0 \leq i \leq d$); (ii) $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq d$); (iii) $I = \sum_{i=0}^d E_i$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. Moreover

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j}.$$

We now define a Leonard system.

Definition 3.1. [26, Definition 1.4] Let d denote a nonnegative integer and let V denote a vector space over \mathbb{K} with dimension $d+1$. By a *Leonard system* on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(v) below.

- (i) Each of A, A^* is a multiplicity-free element in $\text{End}(V)$.
- (ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A .
- (iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^* .
- (iv) $E_i A^* E_j = \begin{cases} 0 & \text{if } |i - j| > 1 \\ \neq 0 & \text{if } |i - j| = 1 \end{cases} \quad (0 \leq i, j \leq d).$
- (v) $E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i - j| > 1 \\ \neq 0 & \text{if } |i - j| = 1 \end{cases} \quad (0 \leq i, j \leq d).$

We refer to d as the *diameter* of Φ , and say Φ is *over* \mathbb{K} . We call V the *vector space underlying* Φ .

We comment on how Leonard pairs and Leonard systems are related. Fix an integer $d \geq 0$ and let V denote a vector space over \mathbb{K} with dimension $d + 1$. Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V . For $0 \leq i \leq d$ let v_i denote a nonzero vector in $E_i V$. Then the sequence $\{v_i\}_{i=0}^d$ is a basis for V that satisfies Definition 2.1(ii). For $0 \leq i \leq d$ let v_i^* denote a nonzero vector in $E_i^* V$. Then the sequence $\{v_i^*\}_{i=0}^d$ is a basis for V that satisfies Definition 2.1(i). By these comments the pair A, A^* is a Leonard pair on V . Conversely let A, A^* denote a Leonard pair on V . By [27, Lemma 3.1] each of A, A^* is multiplicity-free. Let $\{v_i\}_{i=0}^d$ denote a basis for V that satisfies Definition 2.1(ii). For $0 \leq i \leq d$ the vector v_i is an eigenvector for A ; let E_i denote the corresponding primitive idempotent. Let $\{v_i^*\}_{i=0}^d$ denote a basis for V that satisfies Definition 2.1(i). For $0 \leq i \leq d$ the vector v_i^* is an eigenvector for A^* ; let E_i^* denote the corresponding primitive idempotent. Then $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system on V .

Definition 3.2. Referring to the Leonard system Φ from Definition 3.1, for $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with the eigenspace $E_i V$ (resp. $E_i^* V$). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ .

The following notation will be useful. Let λ denote an indeterminate and let $\mathbb{K}[\lambda]$ denote the \mathbb{K} -algebra consisting of the polynomials in λ that have all coefficients in \mathbb{K} .

Definition 3.3. Referring to the Leonard system Φ from Definition 3.1, let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ . For $0 \leq i \leq d$ define polynomials $\tau_i, \eta_i, \tau_i^*, \eta_i^*$ in $\mathbb{K}[\lambda]$ as follows.

$$\begin{aligned}\tau_i &= (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}), \\ \eta_i &= (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1}), \\ \tau_i^* &= (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*), \\ \eta_i^* &= (\lambda - \theta_d^*)(\lambda - \theta_{d-1}^*) \cdots (\lambda - \theta_{d-i+1}^*).\end{aligned}$$

Observe that each of $\tau_i, \eta_i, \tau_i^*, \eta_i^*$ is monic of degree i .

4 The D_4 action

Let Φ denote the Leonard system on V from Definition 3.1. Then each of the following three sequences is a Leonard system on V .

$$\begin{aligned}\Phi^* &:= (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d), \\ \Phi^\downarrow &:= (A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d), \\ \Phi^\Downarrow &:= (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d).\end{aligned}$$

Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all Leonard systems,

$$*^2 = \downarrow^2 = \Downarrow^2 = 1, \tag{2}$$

$$\Downarrow * = * \downarrow, \quad \downarrow * = * \Downarrow, \quad \Downarrow \Downarrow = \Downarrow \downarrow. \tag{3}$$

The group generated by symbols $*, \downarrow, \Downarrow$ subject to the relations (2), (3) is the dihedral group D_4 . We recall D_4 is the group of symmetries of a square, and has 8 elements. Apparently $*, \downarrow, \Downarrow$ induce an action of D_4 on the set of all Leonard systems. Two Leonard systems will be called *relatives* whenever they are in the same orbit of this D_4 action.

For the rest of this paper we will use the following convention.

Definition 4.1. Referring to Leonard system Φ from Definition 3.1, for any element g in the group D_4 and for any object f associated with Φ , let f^g denote the corresponding object for the Leonard system $\Phi^{g^{-1}}$.

5 The standard decomposition and the standard basis

Throughout this section fix an integer $d \geq 0$ and let V denote a vector space over \mathbb{K} with dimension $d+1$. By a *decomposition* of V we mean a sequence $\{V_i\}_{i=0}^d$ of subspaces of V such that V_i has dimension 1 for $0 \leq i \leq d$ and $V = \sum_{i=0}^d V_i$ (direct sum). Let $\{V_i\}_{i=0}^d$ denote a decomposition of V . By the *inversion* of this decomposition we mean the decomposition $\{V_{d-i}\}_{i=0}^d$.

Definition 5.1. Referring to the Leonard system Φ on V from Definition 3.1, observe that $\{E_i^*V\}_{i=0}^d$ is a decomposition of V . We say that this decomposition is Φ -*standard*.

Lemma 5.2. [21, Lemma 5.1] *Referring to the Leonard system Φ on V from Definition 3.1, let v denote a nonzero vector in E_0V . Then for $0 \leq i \leq d$ the element E_i^*v is nonzero and hence a basis for E_i^*V . Moreover the sequence $\{E_i^*v\}_{i=0}^d$ is a basis for V .*

Definition 5.3. [21, Definition 5.2] Referring to the Leonard system Φ on V from Definition 3.1, by a Φ -*standard basis* for V we mean the sequence $\{E_i^*v\}_{i=0}^d$ where v denotes a nonzero vector in E_0V .

Lemma 5.4. *Referring to the Leonard system Φ on V from Definition 3.1, with respect to a Φ -standard basis the matrix representing A is irreducible tridiagonal and the matrix representing A^* is $\text{diag}(\theta_0^*, \theta_1^*, \dots, \theta_d^*)$ where $\{\theta_i^*\}_{i=0}^d$ is the dual eigenvalue sequence of Φ .*

Proof: Immediate from Definition 3.1. □

6 The split decomposition and the split basis

Throughout this section let Φ denote the Leonard system on V from Definition 3.1. For $0 \leq i \leq d$ define

$$U_i = (E_0^*V + E_1^*V + \dots + E_i^*V) \cap (E_iV + E_{i+1}V + \dots + E_dV). \quad (4)$$

By [27, Theorem 20.7] the sequence $\{U_i\}_{i=0}^d$ is a decomposition of V . This decomposition is said to be Φ -*split* [27, Definition 20.2]. By [27, Theorem 20.7] for $0 \leq i \leq d$ both

$$\begin{aligned} U_0 + U_1 + \dots + U_i &= E_0^*V + E_1^*V + \dots + E_i^*V, \\ U_i + U_{i+1} + \dots + U_d &= E_iV + E_{i+1}V + \dots + E_dV. \end{aligned} \quad (5)$$

By [27, Lemma 20.9],

$$(A - \theta_i I)U_i = U_{i+1} \quad (0 \leq i \leq d-1), \quad (A - \theta_d I)U_d = 0, \quad (6)$$

$$(A^* - \theta_i^* I)U_i = U_{i-1} \quad (1 \leq i \leq d), \quad (A^* - \theta_0^* I)U_0 = 0. \quad (7)$$

By (6), (7) for $1 \leq i \leq d$, U_i is invariant under the action of $(A - \theta_{i-1}I)(A^* - \theta_i^*I)$, and the corresponding eigenvalue is a nonzero scalar in \mathbb{K} . We denote this eigenvalue by φ_i . We display a basis for V that illuminates the significance of φ_i . Setting $i = 0$ in (4) we find $U_0 = E_0^*V$. Combining this with (6) we find

$$U_i = (A - \theta_{i-1}I) \cdots (A - \theta_1I)(A - \theta_0I)E_0^*V \quad (0 \leq i \leq d). \quad (8)$$

Let v denote a nonzero vector in E_0^*V . From (8) we find that for $0 \leq i \leq d$ the vector $(A - \theta_{i-1}I) \cdots (A - \theta_0I)v$ is a basis for U_i . By this and since $\{U_i\}_{i=0}^d$ is a decomposition of V we find the sequence

$$(A - \theta_{i-1}I) \cdots (A - \theta_1I)(A - \theta_0I)v \quad (0 \leq i \leq d) \quad (9)$$

is a basis for V . With respect to this basis the matrices representing A and A^* are

$$A : \begin{pmatrix} \theta_0 & & & & \mathbf{0} \\ & 1 & \theta_1 & & \\ & & 1 & \theta_2 & \\ & & & \ddots & \ddots \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \quad A^* : \begin{pmatrix} \theta_0^* & \varphi_1 & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \ddots \\ \mathbf{0} & & & & \varphi_d & \theta_d^* \end{pmatrix}. \quad (10)$$

By a Φ -split basis for V we mean a sequence of the form (9), where v is a nonzero vector in E_0^*V . We call $\{\varphi_i\}_{i=1}^d$ the *first split sequence* of Φ . We let $\{\phi_i\}_{i=1}^d$ denote the first split sequence of Φ^\downarrow and call this the *second split sequence* of Φ . For notational convenience define $\varphi_0 = 0$, $\varphi_{d+1} = 0$, $\phi_0 = 0$, $\phi_{d+1} = 0$.

We now define the parameter array of Φ .

Definition 6.1. [28, Definition 10.1] By the *parameter array* of Φ we mean the sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d)$ where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denotes the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ and $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) denotes the first split sequence (resp. second split sequence) of Φ .

We finish this section with a few characterizations of the Φ -split basis.

Lemma 6.2. [28, Lemma 13.2] *Let $\{v_i\}_{i=0}^d$ denote a sequence of vectors in V , not all zero. Then $\{v_i\}_{i=0}^d$ is a Φ -split basis for V if and only if both (i) $v_0 \in E_0^*V$; (ii) $Av_i = \theta_i v_i + v_{i+1}$ for $0 \leq i \leq d-1$.*

Lemma 6.3. *Let $\{v_i\}_{i=0}^d$ denote a sequence of vectors in V , not all zero. Then $\{v_i\}_{i=0}^d$ is a Φ -split basis for V if and only if both (i) $v_d \in E_d V$; (ii) $A^*v_i = \theta_i^* v_i + \varphi_i v_{i-1}$ for $1 \leq i \leq d$.*

Proof: First assume $\{v_i\}_{i=0}^d$ is a Φ -split basis for V . With respect to $\{v_i\}_{i=0}^d$ the matrices representing A, A^* satisfy (10). Therefore the basis $\{v_i\}_{i=0}^d$ satisfies (i), (ii), and we are done in one direction. To prove the other direction assume $\{v_i\}_{i=0}^d$ satisfies (i), (ii). We will invoke Lemma 6.2. To do this we need to verify that $\{v_i\}_{i=0}^d$ satisfies Lemma 6.2(i), (ii). By (4) we have $E_d V = U_d$. By this and (i) we have $v_d \in U_d$. By this, (ii) and (7) we have $v_i \in U_i$ for $0 \leq i \leq d$. In particular $v_0 \in U_0$. By (4) we have $U_0 = E_0^* V$ so $v_0 \in E_0^* V$. Therefore $\{v_i\}_{i=0}^d$ satisfies Lemma 6.2(i). By (ii) we have $(A - \theta_i I)v_i = \varphi_{i+1}^{-1}(A - \theta_i I)(A^* - \theta_{i+1}^* I)v_{i+1}$ for $0 \leq i \leq d-1$. By this and the discussion below (7) we have $(A - \theta_i I)v_i = v_{i+1}$. Therefore $\{v_i\}_{i=0}^d$ satisfies Lemma 6.2(ii). By Lemma 6.2 the sequence $\{v_i\}_{i=0}^d$ is a Φ -split basis for V . \square

Lemma 6.4. *Let $\{v_i\}_{i=0}^d$ denote a sequence of vectors in V . Then the following are equivalent:*

- (i) *The sequence $\{v_i\}_{i=0}^d$ is a Φ -split basis for V .*
- (ii) *There exists a nonzero $w \in E_d V$ such that*

$$v_i = \frac{(A^* - \theta_{i+1}^* I) \cdots (A^* - \theta_{d-1}^* I)(A^* - \theta_d^* I)w}{\varphi_{i+1} \cdots \varphi_{d-1} \varphi_d} \quad (0 \leq i \leq d).$$

Proof: Immediate from Lemma 6.3. \square

7 A classification of Leonard systems

In [26, Theorem 1.9] Leonard systems are classified up to isomorphism. We now recall this classification.

Theorem 7.1. [26, Theorem 1.9] *Let d denote a nonnegative integer and let*

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d) \quad (11)$$

denote a sequence of scalars taken from \mathbb{K} . There exists a Leonard system Φ over \mathbb{K} with parameter array (11) if and only if the following conditions (PA1)–(PA5) hold.

$$\text{(PA1)} \quad \theta_i \neq \theta_j, \quad \theta_i^* \neq \theta_j^* \quad \text{if } i \neq j, \quad (0 \leq i, j \leq d).$$

$$\text{(PA2)} \quad \varphi_i \neq 0, \quad \phi_i \neq 0 \quad (1 \leq i \leq d).$$

$$\text{(PA3)} \quad \varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \quad (1 \leq i \leq d).$$

$$\text{(PA4)} \quad \phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d).$$

(PA5) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \quad (12)$$

are equal and independent of i for $2 \leq i \leq d-1$.

Moreover, suppose (PA1)–(PA5) hold. Then Φ is unique up to isomorphism of Leonard systems.

Theorem 7.2. [26, Theorem 1.11] *Let Φ denote a Leonard system with parameter array*

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

Then (i)–(iii) hold below.

- (i) *The parameter array of Φ^* is $(\{\theta_i^*\}_{i=0}^d; \{\theta_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d)$.*
- (ii) *The parameter array of Φ^\downarrow is $(\{\theta_i\}_{i=0}^d; \{\theta_{d-i}^*\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d)$.*
- (iii) *The parameter array of Φ^\updownarrow is $(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d)$.*

8 The normalized split basis

Throughout this section let Φ denote the Leonard system on V from Definition 3.1. In an earlier section we discussed the Φ -split basis. For our purpose it is convenient to modify the Φ -split basis by adjusting the normalization.

Lemma 8.1. *Let v denote a nonzero vector in E_0^*V . For $0 \leq i \leq d$ define*

$$u_i = \frac{\tau_i^*(\theta_d^*)(A - \theta_{i-1}I) \cdots (A - \theta_1I)(A - \theta_0I)v}{\varphi_1\varphi_2 \cdots \varphi_i}. \quad (13)$$

Then u_i is a basis for the subspace U_i from line (4). Moreover the sequence $\{u_i\}_{i=0}^d$ is a basis for V .

Proof: Since $\{\theta_j^*\}_{j=0}^d$ are mutually distinct, $\tau_i^*(\theta_d^*) \neq 0$. The first assertion follows from this and the comment below (8). The second assertion follows from this and the fact that $\{U_i\}_{i=0}^d$ is a decomposition of V . \square

Definition 8.2. By a *normalized Φ -split basis* for V we mean a sequence $\{u_i\}_{i=0}^d$ of the form (13), where v is a nonzero vector in E_0^*V .

For the rest of this section we describe the normalized Φ -split basis from various points of view.

Lemma 8.3. *The following (i), (ii) hold.*

(i) Let $\{v_i\}_{i=0}^d$ denote a Φ -split basis for V . Then the sequence

$$\frac{\tau_i^*(\theta_d^*)v_i}{\varphi_1\varphi_2\cdots\varphi_i} \quad (0 \leq i \leq d)$$

is a normalized Φ -split basis for V .

(ii) Let $\{u_i\}_{i=0}^d$ denote a normalized Φ -split basis for V . Then the sequence

$$\frac{\varphi_1\varphi_2\cdots\varphi_i u_i}{\tau_i^*(\theta_d^*)} \quad (0 \leq i \leq d)$$

is a Φ -split basis for V .

Proof: Compare (9) and (13). □

Lemma 8.4. Let $\{u_i\}_{i=0}^d$ denote a sequence of vectors in V . Then the following are equivalent:

- (i) The sequence $\{u_i\}_{i=0}^d$ is a normalized Φ -split basis for V .
- (ii) There exists a nonzero $w \in E_d V$ such that

$$u_i = \tau_i^*(\theta_d^*)(A^* - \theta_{i+1}^* I) \cdots (A^* - \theta_{d-1}^* I)(A^* - \theta_d^* I)w \quad (0 \leq i \leq d).$$

Proof: Immediate from Lemma 6.4 and Lemma 8.3. □

Lemma 8.5. Let $\{u_i\}_{i=0}^d$ denote a sequence of vectors in V , not all zero. Then $\{u_i\}_{i=0}^d$ is a normalized Φ -split basis for V if and only if both (i) $u_0 \in E_0^* V$; (ii) $Au_i = \theta_i^* u_i + \varphi_{i+1}(\theta_d^* - \theta_i^*)^{-1} u_{i+1}$ for $0 \leq i \leq d-1$.

Proof: Immediate from Lemma 8.1. □

Lemma 8.6. Let $\{u_i\}_{i=0}^d$ denote a sequence of vectors in V , not all zero. Then $\{u_i\}_{i=0}^d$ is a normalized Φ -split basis for V if and only if both (i) $u_d \in E_d V$; (ii) $A^* u_i = \theta_i^* u_i + (\theta_d^* - \theta_{i-1}^*) u_{i-1}$ for $1 \leq i \leq d$.

Proof: Immediate from Lemma 8.4. □

Using Lemma 8.5 and Lemma 8.6 we now describe the matrices representing A, A^* with respect to a normalized Φ -split basis for V .

Lemma 8.7. With respect to a normalized Φ -split basis for V , the matrices in $\text{Mat}_{d+1}(\mathbb{K})$ that represent A, A^* are described as follows. The matrix representing A is lower bidiagonal with (i, i) -entry θ_i for $0 \leq i \leq d$ and $(i, i-1)$ -entry $\varphi_i/(\theta_d^* - \theta_{i-1}^*)$ for $1 \leq i \leq d$. The matrix representing A^* is upper bidiagonal with (i, i) -entry θ_i^* for $0 \leq i \leq d$ and $(i-1, i)$ -entry $\theta_d^* - \theta_{i-1}^*$ for $1 \leq i \leq d$. Moreover the matrix representing A^* has constant row sum θ_d^* .

Example 8.8. With reference to Definition 3.1 assume $d = 4$. With respect to a normalized Φ -split basis for V , the matrices representing A, A^* are given below.

$$A : \begin{pmatrix} \theta_0 & 0 & 0 & 0 & 0 \\ \frac{\varphi_1}{\theta_4^* - \theta_0^*} & \theta_1 & 0 & 0 & 0 \\ 0 & \frac{\varphi_2}{\theta_4^* - \theta_1^*} & \theta_2 & 0 & 0 \\ 0 & 0 & \frac{\varphi_3}{\theta_4^* - \theta_2^*} & \theta_3 & 0 \\ 0 & 0 & 0 & \frac{\varphi_4}{\theta_4^* - \theta_3^*} & \theta_4 \end{pmatrix}, \quad A^* : \begin{pmatrix} \theta_0^* & \theta_4^* - \theta_0^* & 0 & 0 & 0 \\ 0 & \theta_1^* & \theta_4^* - \theta_1^* & 0 & 0 \\ 0 & 0 & \theta_2^* & \theta_4^* - \theta_2^* & 0 \\ 0 & 0 & 0 & \theta_3^* & \theta_4^* - \theta_3^* \\ 0 & 0 & 0 & 0 & \theta_4^* \end{pmatrix}.$$

Observe that the matrix representing A^* has constant row sum θ_4^* .

Lemma 8.9. Let $\{u_i\}_{i=0}^d$ denote a sequence of vectors in V , not all zero. Then $\{u_i\}_{i=0}^d$ is a normalized Φ -split basis for V if and only if both (i) $u_i \in U_i$ for $0 \leq i \leq d$; (ii) $\sum_{i=0}^d u_i \in E_d^* V$.

Proof: First assume that $\{u_i\}_{i=0}^d$ is a normalized Φ -split basis for V . The sequence $\{u_i\}_{i=0}^d$ satisfies (i) by Lemma 8.1. By Lemma 8.7 the matrix representing A^* with respect to $\{u_i\}_{i=0}^d$ has constant row sum θ_d^* . Therefore $(A^* - \theta_d^* I) \sum_{i=0}^d u_i = 0$, and thus the sequence $\{u_i\}_{i=0}^d$ satisfies (ii). We have shown that $\{u_i\}_{i=0}^d$ satisfies (i), (ii), and we are done in one direction. To prove the other direction assume $\{u_i\}_{i=0}^d$ satisfies (i), (ii). We will invoke Lemma 8.6. To do this it suffices to verify that $\{u_i\}_{i=0}^d$ satisfies Lemma 8.6(i), (ii). By assumption $u_d \in U_d$. By (4) we have $U_d = E_d V$ so $u_d \in E_d V$. Therefore $\{u_i\}_{i=0}^d$ satisfies Lemma 8.6(i). By (ii) we have

$$\begin{aligned} 0 &= \sum_{i=0}^d (A^* - \theta_d^* I) u_i \\ &= \sum_{i=0}^d (A^* - \theta_i^* I) u_i + \sum_{i=0}^d (\theta_i^* - \theta_d^*) u_i \\ &= \sum_{i=0}^d (A^* - \theta_i^* I) u_i + \sum_{i=1}^d (\theta_{i-1}^* - \theta_d^*) u_{i-1}. \end{aligned}$$

By (i) and (7) we have $(A^* - \theta_i^* I) u_i \in U_{i-1}$ for $1 \leq i \leq d$ and $(A^* - \theta_0^* I) u_0 = 0$. By (i), the comments above and since $\{U_i\}_{i=0}^d$ is a decomposition of V , the sequence $\{u_i\}_{i=0}^d$ satisfies Lemma 8.6(ii). Therefore by Lemma 8.6 the sequence $\{u_i\}_{i=0}^d$ is a normalized Φ -split basis for V . \square

9 The intersection matrix

Throughout this section let Φ denote the Leonard system from Definition 3.1. In this section we recall the intersection matrix of Φ and the dual intersection matrix of Φ .

Definition 9.1. Consider the matrix in $\text{Mat}_{d+1}(\mathbb{K})$ that represents A with respect to a Φ -standard basis. This matrix is irreducible tridiagonal by Lemma 5.4. This matrix will be written as

$$\begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ \mathbf{0} & & & c_d & a_d \end{pmatrix}.$$

We call this matrix the *intersection matrix* of Φ . For notational convenience define $b_d = 0$ and $c_0 = 0$. We call a_i, b_i, c_i ($0 \leq i \leq d$) the *intersection numbers* of Φ .

Definition 9.2. By the *dual intersection matrix* of Φ we mean the intersection matrix for Φ^* . We call a_i^*, b_i^*, c_i^* ($0 \leq i \leq d$) the *dual intersection numbers* of Φ .

Lemma 9.3. [27, Lemma 11.2] *We have $a_i + b_i + c_i = \theta_0$ for $0 \leq i \leq d$.*

We now give explicit formulas for the intersection numbers and the dual intersection numbers. To avoid trivialities assume $d \geq 1$.

Lemma 9.4. [27, Theorem 23.5] *The following (i), (ii) hold.*

$$\begin{aligned} \text{(i)} \quad b_i &= \varphi_{i+1} \frac{\tau_i^*(\theta_i^*)}{\tau_{i+1}^*(\theta_{i+1}^*)} \quad (0 \leq i \leq d-1). \\ \text{(ii)} \quad c_i &= \phi_i \frac{\eta_{d-i}^*(\theta_i^*)}{\eta_{d-i+1}^*(\theta_{i-1}^*)} \quad (1 \leq i \leq d). \end{aligned}$$

Lemma 9.5. [27, Theorem 23.6] *We have*

$$\begin{aligned} a_0 &= \theta_0 + \frac{\varphi_1}{\theta_0^* - \theta_1^*}, \\ a_i &= \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*} \quad (1 \leq i \leq d-1), \\ a_d &= \theta_d + \frac{\varphi_d}{\theta_d^* - \theta_{d-1}^*}. \end{aligned}$$

Lemma 9.6. *The following (i), (ii) hold.*

$$\begin{aligned} \text{(i)} \quad b_i^* &= \varphi_{i+1} \frac{\tau_i(\theta_i)}{\tau_{i+1}(\theta_{i+1})} \quad (0 \leq i \leq d-1). \\ \text{(ii)} \quad c_i^* &= \phi_{d-i+1} \frac{\eta_{d-i}(\theta_i)}{\eta_{d-i+1}(\theta_{i-1})} \quad (1 \leq i \leq d). \end{aligned}$$

Proof: Apply Lemma 9.4 to the Leonard system Φ^* and use Lemma 7.2(i) to obtain the result. \square

Lemma 9.7. *We have*

$$\begin{aligned} a_0^* &= \theta_0^* + \frac{\varphi_1}{\theta_0 - \theta_1}, \\ a_i^* &= \theta_i^* + \frac{\varphi_i}{\theta_i - \theta_{i-1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}} \quad (1 \leq i \leq d-1), \\ a_d^* &= \theta_d^* + \frac{\varphi_d}{\theta_d - \theta_{d-1}}. \end{aligned}$$

Proof: Apply Lemma 9.5 to the Leonard system Φ^* and use Lemma 7.2(i) to obtain the result. \square

10 The tridiagonal relations and the Askey-Wilson relations

Throughout this section let Φ denote the Leonard system from Definition 3.1. We recall the corresponding tridiagonal relations and Askey-Wilson relations.

Lemma 10.1. [12, Theorem 10.1] *There exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ taken from \mathbb{K} such that both*

$$[A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*] = 0, \quad (14)$$

$$[A^*, A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^*A] = 0. \quad (15)$$

The notation $[r, s]$ means $rs - sr$. The sequence is uniquely determined by the Leonard system Φ provided $d \geq 3$.

We call (14), (15) the *tridiagonal relations*.

Lemma 10.2. [30, Corollary 4.4, Theorem 4.5] *Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote scalars in \mathbb{K} . Then these scalars satisfy (14), (15) if and only if the following (i)–(v) hold.*

- (i) $\beta + 1$ is the common value of (12) for $2 \leq i \leq d-1$.
- (ii) $\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1} \quad (1 \leq i \leq d-1)$.
- (iii) $\gamma^* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* \quad (1 \leq i \leq d-1)$.
- (iv) $\varrho = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) \quad (1 \leq i \leq d)$.
- (v) $\varrho^* = \theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) \quad (1 \leq i \leq d)$.

Lemma 10.3. [30, Theorem 1.5] *Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote the scalars from Lemma 10.1. Then there exists a sequence of scalars ω, η, η^* taken from \mathbb{K} such that both*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^*A^2 + \omega A + \eta I, \quad (16)$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^*A = \gamma A^{*2} + \omega A^* + \eta^* I. \quad (17)$$

The sequence is uniquely determined by the Leonard system Φ provided $d \geq 3$.

We call (16), (17) the *Askey-Wilson relations*.

For notational convenience let θ_{-1} and θ_{d+1} (resp. θ_{-1}^* and θ_{d+1}^*) denote the scalars in \mathbb{K} which satisfy Lemma 10.2(ii) (resp. Lemma 10.2(iii)) for $i = 0$ and $i = d$.

Lemma 10.4. [30, Corollary 5.2, Theorem 5.3] *Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote the scalars from Lemma 10.1. Let ω, η, η^* denote scalars in \mathbb{K} . Then ω, η, η^* satisfy (16), (17) if and only if the following (i)–(iv) hold.*

$$(i) \quad \omega = a_i(\theta_i^* - \theta_{i+1}^*) + a_{i-1}(\theta_{i-1}^* - \theta_{i-2}^*) - \gamma(\theta_i^* + \theta_{i-1}^*) \quad (1 \leq i \leq d).$$

$$(ii) \quad \omega = a_i^*(\theta_i - \theta_{i+1}) + a_{i-1}^*(\theta_{i-1} - \theta_{i-2}) - \gamma^*(\theta_i + \theta_{i-1}) \quad (1 \leq i \leq d).$$

$$(iii) \quad \eta = a_i^*(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) - \gamma^*\theta_i^2 - \omega\theta_i \quad (0 \leq i \leq d).$$

$$(iv) \quad \eta^* = a_i(\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) - \gamma\theta_i^{*2} - \omega\theta_i^* \quad (0 \leq i \leq d).$$

11 Leonard systems of dual q -Krawtchouk type

For the past few sections we have been discussing general Leonard systems. We now consider a family of Leonard systems said to have dual q -Krawtchouk type.

For the rest of the paper let q denote a nonzero scalar in \mathbb{K} such that $q^2 \neq 1$.

We will be discussing the Leonard system Φ from Definition 3.1. Let $\overline{\mathbb{K}}$ denote the algebraic closure of \mathbb{K} .

Definition 11.1. [27, Example 35.8] The Leonard system Φ is said to have *dual q -Krawtchouk type* whenever there exist scalars $h, h^*, \kappa, \kappa^*, v$ in $\overline{\mathbb{K}}$ such that κ, κ^*, v are nonzero and both

$$\theta_i = h + \kappa q^{d-2i} + v q^{2i-d}, \quad (18)$$

$$\theta_i^* = h^* + \kappa^* q^{d-2i} \quad (19)$$

for $0 \leq i \leq d$ and both

$$\varphi_i = \kappa \kappa^* q^{d+1-2i} (q^i - q^{-i})(q^{i-d-1} - q^{d+1-i}), \quad (20)$$

$$\phi_i = \kappa^* v q^{d+1-2i} (q^i - q^{-i})(q^{i-d-1} - q^{d+1-i}) \quad (21)$$

for $1 \leq i \leq d$.

For the rest of this section assume Φ has dual q -Krawtchouk type with the scalars $h, h^*, \kappa, \kappa^*, v$ as in Definition 11.1.

Lemma 11.2. *The following (i), (ii) hold for $0 \leq i, j \leq d$.*

$$(i) \quad \theta_i - \theta_j = q^d (q^{-i} - q^{-j})(q^{-i} + q^{-j})(\kappa - v q^{2i+2j-2d}).$$

$$(ii) \quad \theta_i^* - \theta_j^* = \kappa^* q^d (q^{-i} - q^{-j})(q^{-i} + q^{-j}).$$

Proof: Immediate from (18) and (19). \square

Lemma 11.3. *The following (i), (ii) hold.*

$$(i) \quad q^{2i} \neq 1 \quad (1 \leq i \leq d).$$

$$(ii) \quad \kappa \neq vq^{2i-2d} \quad (1 \leq i \leq 2d-1).$$

Proof: By Lemma 11.2(i) and since $\{\theta_i\}_{i=0}^d$ are mutually distinct. \square

Lemma 11.4. *The intersection numbers of Φ satisfy the following (i)–(iii).*

$$(i) \quad b_i = \kappa q^i (q^{d-i} - q^{i-d}) \quad (0 \leq i \leq d-1).$$

$$(ii) \quad c_i = vq^{i-d} (q^{-i} - q^i) \quad (1 \leq i \leq d).$$

$$(iii) \quad a_i = h + (\kappa + v)q^{2i-d} \quad (0 \leq i \leq d).$$

Proof: In the equations of Lemma 9.4 and Lemma 9.5, evaluate the right-hand sides using Definition 11.1 and Lemma 11.2(ii), and then simplify the result. \square

Lemma 11.5. *The dual intersection numbers of Φ satisfy the following (i)–(iii).*

$$(i) \quad b_i^* = \frac{\kappa \kappa^* q^i (q^{d-i} - q^{i-d})(\kappa - vq^{2i-2d})}{(\kappa - vq^{4i-2d})(\kappa - vq^{4i-2d+2})} \quad (1 \leq i \leq d-1), \quad b_0^* = \frac{\kappa \kappa^* (q^d - q^{-d})}{\kappa - vq^{2-2d}}.$$

$$(ii) \quad c_i^* = \frac{v \kappa^* q^{5i-3d-2} (q^{-i} - q^i)(\kappa - vq^{2i})}{(\kappa - vq^{4i-2d})(\kappa - vq^{4i-2d-2})} \quad (1 \leq i \leq d-1), \quad c_d^* = \frac{v \kappa^* q^{2d-2} (q^{-d} - q^d)}{\kappa - vq^{2d-2}}.$$

$$(iii) \quad a_i^* = \theta_0^* - b_i^* - c_i^* \quad (0 \leq i \leq d).$$

Proof: (i), (ii) In the equations of Lemma 9.6, evaluate the right-hand sides using Definition 11.1 and Lemma 11.2(i), and then simplify the result.

(iii) Apply Lemma 9.3 to the Leonard system Φ^* . \square

Lemma 11.6. *Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote the scalars from \mathbb{K} which satisfy the following (i)–(v).*

$$(i) \quad \beta = q^2 + q^{-2}.$$

$$(ii) \quad \gamma = h(2 - \beta).$$

$$(iii) \quad \gamma^* = h^*(2 - \beta).$$

$$(iv) \quad \varrho = h^2(\beta - 2) - \kappa v(\beta^2 - 4).$$

$$(v) \quad \varrho^* = h^{*2}(\beta - 2).$$

Then $\beta, \gamma, \gamma^, \varrho, \varrho^*$ satisfy (14), (15).*

Proof: Routine verification using Lemma 10.2 and Definition 11.1. \square

Lemma 11.7. *Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote the scalars from Lemma 11.6. Let ω, η, η^* denote the scalars from \mathbb{K} which satisfy the following (i)–(iii).*

- (i) $\omega = (\beta - 2)(2hh^* - (\kappa + v)\kappa^*)$.
- (ii) $\eta = \kappa v h^*(\beta^2 - 4) + \kappa v \kappa^*(q + q^{-1})(q - q^{-1})^2(q^{d+1} + q^{-d-1}) - h(\beta - 2)(hh^* - (\kappa + v)\kappa^*)$.
- (iii) $\eta^* = h^*(\beta - 2)((\kappa + v)\kappa^* - hh^*)$.

Then $\beta, \gamma, \gamma^, \varrho, \varrho^*, \omega, \eta, \eta^*$ satisfy (16), (17).*

Proof: In the equations of Lemma 10.4, evaluate the right-hand sides using Definition 11.1, Lemma 11.4, Lemma 11.5 and Lemma 11.6, and then simplify the result. \square

Note 11.8. Among the relatives of Φ we find that the Leonard system Φ^\Downarrow also has dual q -Krawtchouk type with $h^\Downarrow = h, (h^*)^\Downarrow = h^*, \kappa^\Downarrow = v, (\kappa^*)^\Downarrow = \kappa^*, v^\Downarrow = \kappa$.

12 The algebra $U_q(\mathfrak{sl}_2)$

In the previous section we discussed Leonard systems of dual q -Krawtchouk type. We now turn our attention to the algebra $U_q(\mathfrak{sl}_2)$. Later we will relate the Leonard systems of dual q -Krawtchouk type and the algebra $U_q(\mathfrak{sl}_2)$.

Definition 12.1. Let $U_q(\mathfrak{sl}_2)$ denote the \mathbb{K} -algebra with generators $k^{\pm 1}, e, f$ and relations

$$\begin{aligned} kk^{-1} &= k^{-1}k = 1, \\ ke &= q^2ek, \quad kf = q^{-2}fk, \\ ef - fe &= \frac{k - k^{-1}}{q - q^{-1}}. \end{aligned}$$

Lemma 12.2. [15, Theorem 2.1] *The algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to the \mathbb{K} -algebra with generators $x, y, z^{\pm 1}$ and relations*

$$zz^{-1} = z^{-1}z = 1, \tag{22}$$

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \tag{23}$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \tag{24}$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1. \tag{25}$$

An isomorphism with the presentation in Definition 12.1 is given by

$$\begin{aligned} z^{\pm 1} &\rightarrow k^{\pm 1}, \\ y &\rightarrow k^{-1} - q(q - q^{-1})k^{-1}e, \\ x &\rightarrow k^{-1} + (q - q^{-1})f. \end{aligned}$$

The inverse of this isomorphism is given by

$$\begin{aligned} k^{\pm 1} &\rightarrow z^{\pm 1}, \\ f &\rightarrow (q - q^{-1})^{-1}(x - z^{-1}), \\ e &\rightarrow q^{-1}(q - q^{-1})^{-1}(1 - zy). \end{aligned}$$

We call $x, y, z^{\pm 1}$ the *equitable generators* for $U_q(\mathfrak{sl}_2)$. From now on we identify the versions of $U_q(\mathfrak{sl}_2)$ in Definition 12.1 and Lemma 12.2 via the isomorphism in Lemma 12.2.

We now discuss finite-dimensional $U_q(\mathfrak{sl}_2)$ -modules. For an integer n we define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Lemma 12.3. [16, Theorem 2.6] *For an integer $d \geq 0$ and for $\epsilon \in \{-1, 1\}$ there exists a $U_q(\mathfrak{sl}_2)$ -module $L(d, \epsilon)$ with the following properties. $L(d, \epsilon)$ has a basis $\{v_i\}_{i=0}^d$ such that*

$$\begin{aligned} kv_i &= \epsilon q^{d-2i} v_i & (0 \leq i \leq d), \\ fv_i &= [i+1]_q v_{i+1} & (0 \leq i \leq d-1), \quad fv_d = 0, \\ ev_i &= \epsilon [d-i+1]_q v_{i-1} & (1 \leq i \leq d), \quad ev_0 = 0. \end{aligned}$$

The $U_q(\mathfrak{sl}_2)$ -module $L(d, \epsilon)$ is irreducible provided that $q^{2i} \neq 1$ for $1 \leq i \leq d$. Assume $q^{2i} \neq 1$ for $1 \leq i \leq d$. Then every irreducible $U_q(\mathfrak{sl}_2)$ -module of dimension $d+1$ is isomorphic to either $L(d, 1)$ or $L(d, -1)$.

Lemma 12.4. [15, Lemma 4.2] *For an integer $d \geq 0$ and for $\epsilon \in \{-1, 1\}$, the $U_q(\mathfrak{sl}_2)$ -module $L(d, \epsilon)$ has a basis $\{u_i\}_{i=0}^d$ such that*

$$\begin{aligned} \epsilon x u_i &= q^{d-2i} u_i & (0 \leq i \leq d), \\ (\epsilon y - q^{2i-d}) u_i &= (q^{-d} - q^{2i+2-d}) u_{i+1} & (0 \leq i \leq d-1), \quad (\epsilon y - q^d) u_d = 0, \\ (\epsilon z - q^{2i-d}) u_i &= (q^d - q^{2i-2-d}) u_{i-1} & (1 \leq i \leq d), \quad (\epsilon z - q^{-d}) u_0 = 0. \end{aligned}$$

Definition 12.5. We call the basis $\{u_i\}_{i=0}^d$ from Lemma 12.4 a *normalized x -eigenbasis* for $L(d, \epsilon)$. Observe that this basis satisfies $\epsilon y u = q^{-d} u$ and $\epsilon z u = q^d u$ where $u = \sum_{i=0}^d u_i$.

Lemma 12.6. [15, Lemma 4.2] *For an integer $d \geq 0$ and for $\epsilon \in \{-1, 1\}$, the $U_q(\mathfrak{sl}_2)$ -module $L(d, \epsilon)$ has a basis $\{u_i\}_{i=0}^d$ such that*

$$\epsilon y u_i = q^{d-2i} u_i \quad (0 \leq i \leq d), \tag{26}$$

$$(\epsilon z - q^{2i-d}) u_i = (q^{-d} - q^{2i+2-d}) u_{i+1} \quad (0 \leq i \leq d-1), \quad (\epsilon z - q^d) u_d = 0, \tag{27}$$

$$(\epsilon x - q^{2i-d}) u_i = (q^d - q^{2i-2-d}) u_{i-1} \quad (1 \leq i \leq d), \quad (\epsilon x - q^{-d}) u_0 = 0. \tag{28}$$

Definition 12.7. We call the basis $\{u_i\}_{i=0}^d$ from Lemma 12.6 a *normalized y -eigenbasis* for $L(d, \epsilon)$. Observe that this basis satisfies $\epsilon z u = q^{-d} u$ and $\epsilon x u = q^d u$ where $u = \sum_{i=0}^d u_i$.

Lemma 12.8. [15, Lemma 4.2] *For an integer $d \geq 0$ and for $\epsilon \in \{-1, 1\}$, the $U_q(\mathfrak{sl}_2)$ -module $L(d, \epsilon)$ has a basis $\{u_i\}_{i=0}^d$ such that*

$$\begin{aligned} \epsilon z u_i &= q^{d-2i} u_i & (0 \leq i \leq d), \\ (\epsilon x - q^{2i-d}) u_i &= (q^{-d} - q^{2i+2-d}) u_{i+1} & (0 \leq i \leq d-1), \quad (\epsilon x - q^d) u_d = 0, \\ (\epsilon y - q^{2i-d}) u_i &= (q^d - q^{2i-2-d}) u_{i-1} & (1 \leq i \leq d), \quad (\epsilon y - q^{-d}) u_0 = 0. \end{aligned}$$

Definition 12.9. We call the basis $\{u_i\}_{i=0}^d$ from Lemma 12.8 a *normalized z -eigenbasis* for $L(d, \epsilon)$. Observe that this basis satisfies $\epsilon x u = q^{-d} u$ and $\epsilon y u = q^d u$ where $u = \sum_{i=0}^d u_i$.

Definition 12.10. [16, p. 21] Let

$$\Delta = ef + \frac{q^{-1}k + qk^{-1}}{(q - q^{-1})^2}.$$

We call Δ the *Casimir element* of $U_q(\mathfrak{sl}_2)$.

Lemma 12.11. [16, Lemma 2.7] *The element Δ is central in $U_q(\mathfrak{sl}_2)$.*

Lemma 12.12. [16, Lemma 2.7] *For an integer $d \geq 0$ and for $\epsilon \in \{-1, 1\}$, the element Δ acts on the $U_q(\mathfrak{sl}_2)$ -module $L(d, \epsilon)$ as the identity times*

$$\epsilon \frac{q^{d+1} + q^{-d-1}}{(q - q^{-1})^2}. \quad (29)$$

For the rest of this section suppose q is not a root of unity and $\text{char}(\mathbb{K}) \neq 2$.

Lemma 12.13. [16, Lemma 2.7] *For all integers $d \geq 0$ and for all $\epsilon \in \{-1, 1\}$, the scalars (29) are mutually distinct.*

Let M denote a finite-dimensional $U_q(\mathfrak{sl}_2)$ -module. For an integer $d \geq 0$ and for $\epsilon \in \{-1, 1\}$ let $M_{d,\epsilon}$ denote the subspace of M spanned by the irreducible $U_q(\mathfrak{sl}_2)$ -submodules of M which are isomorphic to $L(d, \epsilon)$. Observe that $M_{d,\epsilon}$ is a $U_q(\mathfrak{sl}_2)$ -submodule of M . We call $M_{d,\epsilon}$ the *homogeneous component of M associated with d and ϵ* . By [16, p. 22] the homogeneous component $M_{d,\epsilon}$ is the eigenspace for Δ associated with eigenvalue (29). Moreover by [16, p. 22],

$$M = \sum M_{d,\epsilon} \quad (\text{direct sum}), \quad (30)$$

where the sum is over all integers $d \geq 0$ and $\epsilon \in \{-1, 1\}$.

We emphasize one point for later use.

Lemma 12.14. [16, Theorem 2.9] *Every finite-dimensional $U_q(\mathfrak{sl}_2)$ -module is semisimple.*

13 $U_q(\mathfrak{sl}_2)$ and Leonard systems of dual q -Krawtchouk type

In this section we display two $U_q(\mathfrak{sl}_2)$ -module structures associated with a given Leonard system of dual q -Krawtchouk type. Prior to this display we make some comments. Throughout this section let V denote a vector space over \mathbb{K} with finite positive dimension. Let $A : V \rightarrow V$ denote a linear transformation. For $\theta \in \mathbb{K}$ define $V_A(\theta) = \{v \in V \mid Av = \theta v\}$.

Lemma 13.1. [14, Lemma 6.2] *Let $A : V \rightarrow V$ and $B : V \rightarrow V$ denote linear transformations. Then for all nonzero $\theta \in \mathbb{K}$ the following are equivalent:*

- (i) *The expression $qAB - q^{-1}BA - (q - q^{-1})I$ vanishes on $V_A(\theta)$.*
- (ii) *$(B - \theta^{-1}I)V_A(\theta) \subseteq V_A(q^{-2}\theta)$.*

Lemma 13.2. [14, Lemma 6.3] *Let $A : V \rightarrow V$ and $B : V \rightarrow V$ denote linear transformations. Then for all nonzero $\theta \in \mathbb{K}$ the following are equivalent:*

- (i) *The expression $qAB - q^{-1}BA - (q - q^{-1})I$ vanishes on $V_B(\theta)$.*
- (ii) *$(A - \theta^{-1}I)V_B(\theta) \subseteq V_B(q^2\theta)$.*

For the rest of this section let Φ denote a Leonard system on V as in Definition 3.1. Assume Φ has dual q -Krawtchouk type. Let $h, h^*, \kappa, \kappa^*, v$ denote the corresponding parameters from Definition 11.1. We are going to define something that turns out to be unique up to sign. For notational convenience we fix $\epsilon \in \{1, -1\}$.

Definition 13.3. Let $\{X_i\}_{i=0}^d$ denote the Φ^\downarrow -split decomposition of V . Let $\{Y_i\}_{i=0}^d$ denote the inverted Φ -split decomposition of V . Let $\{Z_i\}_{i=0}^d$ denote the Φ -standard decomposition of V . Define the linear transformations X, Y, Z in $\text{End}(V)$ as follows.

$$(\epsilon X - q^{d-2i}I)X_i = 0 \quad (0 \leq i \leq d), \quad (31)$$

$$(\epsilon Y - q^{d-2i}I)Y_i = 0 \quad (0 \leq i \leq d), \quad (32)$$

$$(\epsilon Z - q^{d-2i}I)Z_i = 0 \quad (0 \leq i \leq d). \quad (33)$$

Lemma 13.4. *There exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on V such that the equitable generators x, y, z of $U_q(\mathfrak{sl}_2)$ act on V as X, Y, Z , respectively.*

Proof: By construction Z is invertible. We show that X, Y, Z, Z^{-1} satisfy the defining relations (22)–(25) of $U_q(\mathfrak{sl}_2)$. By construction Z, Z^{-1} satisfy (22). Next we show that Y, Z satisfy (24). Since $\{Z_i\}_{i=0}^d$ is the Φ -standard decomposition of V , we have $(A^* - \theta_i^*I)Z_i = 0$ for $0 \leq i \leq d$. Using (19) to compare this with (33) we find

$$\epsilon Z = \kappa^{*-1}(A^* - h^*I). \quad (34)$$

By (7) and since $\{Y_i\}_{i=0}^d$ is the inverted Φ -split decomposition of V , we have $(A^* - \theta_{d-i}^*I)Y_i = Y_{i+1}$ for $0 \leq i \leq d-1$ and $(A^* - \theta_0^*I)Y_d = 0$. By this, (19) and (34) we have $(\epsilon Z - q^{2i-d}I)Y_i = Y_{i+1}$ for $0 \leq i \leq d-1$ and $(\epsilon Z - q^dI)Y_d = 0$. By this, Lemma 13.1 and since $\{Y_i\}_{i=0}^d$ is

a decomposition of V , we obtain that Y, Z satisfy (24). Next we show that Z, X satisfy (25). By applying (7) to Φ^\downarrow and since $\{X_i\}_{i=0}^d$ is the Φ^\downarrow -split decomposition of V , we have $(A^* - \theta_{d-i}^* I)X_i = X_{i-1}$ for $1 \leq i \leq d$ and $(A^* - \theta_d^* I)X_0 = 0$. By this, (19) and (34) we have $(\epsilon Z - q^{2i-d} I)X_i = X_{i-1}$ for $1 \leq i \leq d$ and $(\epsilon Z - q^{-d} I)X_0 = 0$. By this, Lemma 13.2 and since $\{X_i\}_{i=0}^d$ is a decomposition of V , we obtain that Z, X satisfy (25). Next we show that X, Y satisfy (23). By construction $Z_i = E_i^* V$ for $0 \leq i \leq d$. By (5) and since $\{Y_i\}_{i=0}^d$ is the inverted Φ -split decomposition of V , we have $E_i V + E_{i+1} V + \cdots + E_d V = Y_0 + Y_1 + \cdots + Y_{d-i}$ for $0 \leq i \leq d$. Now for $0 \leq i \leq d$ we have

$$\begin{aligned} X_i &= (E_d^* V + E_{d-1}^* V + \cdots + E_{d-i}^* V) \cap (E_i V + E_{i+1} V + \cdots + E_d V) \\ &= (Z_d + Z_{d-1} + \cdots + Z_{d-i}) \cap (Y_0 + Y_1 + \cdots + Y_{d-i}). \end{aligned} \quad (35)$$

By (24) and Lemma 13.2, we have $(\epsilon Y - q^{2i-d} I)Z_i \subseteq Z_{i-1}$ for $1 \leq i \leq d$ and $(\epsilon Y - q^{-d} I)Z_0 = 0$. Combining this with (32) and (35), we have $(\epsilon Y - q^{2i-d} I)X_i \subseteq X_{i+1}$ for $0 \leq i \leq d-1$ and $(\epsilon Y - q^d I)X_d = 0$. By this, Lemma 13.1 and since $\{X_i\}_{i=0}^d$ is a decomposition of V , we obtain X, Y satisfy (23). We have now shown that X, Y, Z, Z^{-1} satisfy the relations (22)–(25). Therefore there exists a $U_q(\mathfrak{sl}_2)$ -module structure on V such that x, y, z act on V as X, Y, Z , respectively. This $U_q(\mathfrak{sl}_2)$ -module structure is unique since $x, y, z^{\pm 1}$ generate $U_q(\mathfrak{sl}_2)$. \square

We now display the second $U_q(\mathfrak{sl}_2)$ -module structure on V .

Definition 13.5. Let $\{X_i^\downarrow\}_{i=0}^d$ denote the $\Phi^{\downarrow\downarrow}$ -split decomposition of V . Let $\{Y_i^\downarrow\}_{i=0}^d$ denote the inverted Φ^\downarrow -split decomposition of V . Define the linear transformations $X^\downarrow, Y^\downarrow$ in $\text{End}(V)$ as follows.

$$\begin{aligned} (\epsilon X^\downarrow - q^{d-2i} I)X_i^\downarrow &= 0 & (0 \leq i \leq d), \\ (\epsilon Y^\downarrow - q^{d-2i} I)Y_i^\downarrow &= 0 & (0 \leq i \leq d). \end{aligned}$$

Lemma 13.6. *There exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on V such that the equitable generators x, y, z of $U_q(\mathfrak{sl}_2)$ act on V as $X^\downarrow, Y^\downarrow, Z$, respectively.*

Proof: Recall that the Leonard system Φ^\downarrow has dual q -Krawtchouk type. Apply Lemma 13.4 to Φ^\downarrow . \square

Lemma 13.7. *The $U_q(\mathfrak{sl}_2)$ -module V from Lemma 13.4 is isomorphic to $L(d, \epsilon)$.*

Proof: By construction Z is diagonalizable on V . By this and [16, Theorem 2.9] we find that V is a direct sum of irreducible $U_q(\mathfrak{sl}_2)$ -submodules of V . Observe that Z is diagonalizable on each $U_q(\mathfrak{sl}_2)$ -module in the sum. By construction ϵq^d is an eigenvalue of Z . By these comments there exists a $U_q(\mathfrak{sl}_2)$ -submodule W of V in the sum such that ϵq^d is an eigenvalue for Z on W . By Lemma 11.3(i) and Lemma 12.8 any irreducible $U_q(\mathfrak{sl}_2)$ -module that has ϵq^d as an eigenvalue for Z has dimension at least $d+1$. Therefore $V = W$. By Lemma 12.3 the $U_q(\mathfrak{sl}_2)$ -module V is isomorphic to $L(d, \epsilon)$. \square

Lemma 13.8. *The $U_q(\mathfrak{sl}_2)$ -module V from Lemma 13.6 is isomorphic to $L(d, \epsilon)$.*

Proof: Recall that the Leonard system Φ^\Downarrow has dual q -Krawtchouk type. Apply Lemma 13.7 to Φ^\Downarrow . \square

Lemma 13.9. *For the $U_q(\mathfrak{sl}_2)$ -module structure on V from Lemma 13.4, the following coincide:*

- (i) *The inversion of a normalized Φ -split basis for V .*
- (ii) *A normalized y -eigenbasis for the $U_q(\mathfrak{sl}_2)$ -module V .*

Proof: Let $\{u_i\}_{i=0}^d$ denote a basis for V . Recall the Φ -split decomposition $\{U_i\}_{i=0}^d$ of V from (4). By construction $Y_i = U_{d-i}$ ($0 \leq i \leq d$) and $Z_d = E_d^*V$. By this and Lemma 8.9 the sequence $\{u_i\}_{i=0}^d$ is the inversion of a normalized Φ -split basis for V if and only if $u_i \in Y_i$ ($0 \leq i \leq d$) and $(\epsilon Z - q^{-d}I) \sum_{i=0}^d u_i = 0$. By comparing this to (26) and the comment in Definition 12.7, the sequence $\{u_i\}_{i=0}^d$ is the inversion of a normalized Φ -split basis for V if and only if it is a normalized Y -eigenbasis for the $U_q(\mathfrak{sl}_2)$ -module V . \square

Lemma 13.10. *For the $U_q(\mathfrak{sl}_2)$ -module structure on V from Lemma 13.6, the following coincide:*

- (i) *The inversion of a normalized Φ^\Downarrow -split basis for V .*
- (ii) *A normalized y -eigenbasis for the $U_q(\mathfrak{sl}_2)$ -module V .*

Proof: Recall that the Leonard system Φ^\Downarrow has dual q -Krawtchouk type. Apply Lemma 13.9 to Φ^\Downarrow . \square

Recall our Leonard system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$. We now show how A, A^* are related to the maps X, Y, Z from Definition 13.3.

Lemma 13.11. *The maps A, A^* can be expressed in terms of X, Y, Z as follows.*

$$A = h1 + \epsilon\kappa X + \epsilon vY, \tag{36}$$

$$A^* = h^*1 + \epsilon\kappa^* Z. \tag{37}$$

Proof: Line (37) follows from (34). It remains to show that (36) holds. Let $\{u_i\}_{i=0}^d$ denote the inversion of a normalized Φ -split basis for V . By Lemma 13.9 this is also a normalized Y -eigenbasis for the $U_q(\mathfrak{sl}_2)$ -module V from Lemma 13.4. Now compare the action of each side of (36) on the basis $\{u_i\}_{i=0}^d$ as follows. Let i be given. On the left-hand side evaluate Au_i using Lemma 8.7. On the right-hand side evaluate $(h1 + \epsilon\kappa X + \epsilon vY)u_i$ using (26), (28). By comparing the result using (18), (20) and Lemma 11.2(ii), we find that both sides are equal. Therefore (36) holds. \square

Next we show how A, A^* are related to the maps $X^\Downarrow, Y^\Downarrow$ from Definition 13.5 and the map Z from Definition 13.3.

Lemma 13.12. *The maps A, A^* can be expressed in terms of $X^\downarrow, Y^\downarrow, Z$ as follows.*

$$\begin{aligned} A &= h1 + \epsilon\kappa Y^\downarrow + \epsilon v X^\downarrow, \\ A^* &= h^*1 + \epsilon\kappa^* Z. \end{aligned}$$

Proof: Recall that the Leonard system Φ^\downarrow has dual q -Krawtchouk type. Apply Lemma 13.11 to Φ^\downarrow and use Note 11.8 to obtain the result. \square

We now express X, Y, Z in terms of A, A^* . The expressions will involve the inverse of $A^* - h^*I$. We take a moment to verify that the inverse exists.

Lemma 13.13. *The eigenvalues of $A^* - h^*I$ are $\kappa^* q^{d-2i}$ ($0 \leq i \leq d$). Moreover $A^* - h^*I$ is invertible.*

Proof: By construction $A^* - h^*I$ is diagonalizable with eigenvalues $\theta_i^* - h^*$ ($0 \leq i \leq d$). By (19) we have $\theta_i^* - h^* = \kappa^* q^{d-2i}$ ($0 \leq i \leq d$) so these scalars are all nonzero. By these comments $A^* - h^*I$ is invertible. \square

Lemma 13.14. *The maps X, Y, Z can be expressed in terms of A, A^* as follows.*

$$\epsilon X = \frac{qB - q^{-1}B^*BB^{*-1}}{\kappa q^{-1}(q^2 - q^{-2})} + \frac{\kappa^*(\kappa q^{-1} - vq)B^{*-1}}{\kappa(q + q^{-1})} \quad (38)$$

$$= \frac{qB^{*-1}BB^* - q^{-1}B}{\kappa q(q^2 - q^{-2})} + \frac{\kappa^*(\kappa q - vq^{-1})B^{*-1}}{\kappa(q + q^{-1})}, \quad (39)$$

$$\epsilon Y = \frac{qB - q^{-1}B^{*-1}BB^*}{vq^{-1}(q^2 - q^{-2})} + \frac{\kappa^*(vq^{-1} - \kappa q)B^{*-1}}{v(q + q^{-1})} \quad (40)$$

$$= \frac{qB^*BB^{*-1} - q^{-1}B}{vq(q^2 - q^{-2})} + \frac{\kappa^*(vq - \kappa q^{-1})B^{*-1}}{v(q + q^{-1})}, \quad (41)$$

$$\epsilon Z = \kappa^{*-1}B^*, \quad (42)$$

where $B = A - hI$ and $B^* = A^* - h^*I$.

Proof: By Lemma 13.11,

$$B = \epsilon\kappa X + \epsilon v Y, \quad B^* = \epsilon\kappa^* Z. \quad (43)$$

Thus (42) holds. Next we show that (38), (39) hold. To do this we claim

$$\kappa\kappa^* \frac{qXZ - q^{-1}ZX}{q - q^{-1}} + v\kappa^*I = \frac{qBB^* - q^{-1}B^*B}{q - q^{-1}}. \quad (44)$$

To prove the claim we evaluate B, B^* using (43) and simplify the result using (24). By (25),

$$\frac{qZX - q^{-1}XZ}{q - q^{-1}} = I. \quad (45)$$

Now view (44) and (45) as a system of linear equations in XZ and ZX . Solving the system we have

$$XZ = \frac{q^2 BB^* - B^* B}{\kappa \kappa^* (q^2 - q^{-2})} + \frac{\kappa q^{-1} - vq}{\kappa (q + q^{-1})} I, \quad (46)$$

$$ZX = \frac{BB^* - q^{-2} B^* B}{\kappa \kappa^* (q^2 - q^{-2})} + \frac{\kappa q - vq^{-1}}{\kappa (q + q^{-1})} I. \quad (47)$$

In (46) multiply each side on the right by Z^{-1} and evaluate Z using (42) to obtain (38). In (47) multiply each side on the left by Z^{-1} and evaluate Z using (42) to obtain (39). Next we show that (40), (41) hold. To do this we claim

$$\kappa^* v \frac{qZY - q^{-1}YZ}{q - q^{-1}} + \kappa \kappa^* I = \frac{qB^* B - q^{-1} BB^*}{q - q^{-1}}. \quad (48)$$

To prove the claim we evaluate B, B^* using (43) and simplify the result using (25). By (24),

$$\frac{qYZ - q^{-1}ZY}{q - q^{-1}} = I. \quad (49)$$

Now view (48) and (49) as a system of linear equations in ZY and YZ . Solving the system we have

$$ZY = \frac{q^2 B^* B - BB^*}{v \kappa^* (q^2 - q^{-2})} + \frac{vq^{-1} - \kappa q}{v(q + q^{-1})} I, \quad (50)$$

$$YZ = \frac{B^* B - q^{-2} BB^*}{v \kappa^* (q^2 - q^{-2})} + \frac{vq - \kappa q^{-1}}{v(q + q^{-1})} I. \quad (51)$$

In (50) multiply each side on the left by Z^{-1} and evaluate Z using (42) to obtain (40). In (51) multiply each side on the right by Z^{-1} and evaluate Z using (42) to obtain (41). \square

Next we express $X^\downarrow, Y^\downarrow$ in terms of A, A^*

Lemma 13.15. *The maps $X^\downarrow, Y^\downarrow$ can be expressed in terms of A, A^* as follows.*

$$\begin{aligned} \epsilon X^\downarrow &= \frac{qB - q^{-1} B^* BB^{*-1}}{vq^{-1}(q^2 - q^{-2})} + \frac{\kappa^*(vq^{-1} - \kappa q)B^{*-1}}{v(q + q^{-1})} \\ &= \frac{qB^{*-1}BB^* - q^{-1}B}{vq(q^2 - q^{-2})} + \frac{\kappa^*(vq - \kappa q^{-1})B^{*-1}}{v(q + q^{-1})}, \end{aligned} \quad (52)$$

$$\begin{aligned} \epsilon Y^\downarrow &= \frac{qB - q^{-1} B^{*-1} BB^*}{\kappa q^{-1}(q^2 - q^{-2})} + \frac{\kappa^*(\kappa q^{-1} - vq)B^{*-1}}{\kappa(q + q^{-1})} \\ &= \frac{qB^* BB^{*-1} - q^{-1}B}{\kappa q(q^2 - q^{-2})} + \frac{\kappa^*(\kappa q - vq^{-1})B^{*-1}}{\kappa(q + q^{-1})}, \end{aligned} \quad (53)$$

where $B = A - hI$ and $B^* = A^* - h^*I$.

Proof: Recall that the Leonard system Φ^\downarrow has dual q -Krawtchouk type. Apply Lemma 13.14 to Φ^\downarrow and use Note 11.8 to obtain the result. \square

We comment on how the two $U_q(\mathfrak{sl}_2)$ -module structures are related.

Lemma 13.16. *The $U_q(\mathfrak{sl}_2)$ -module structure from Lemma 13.4 and the $U_q(\mathfrak{sl}_2)$ -module structure from Lemma 13.6 are related as follows.*

$$X^\Downarrow = \kappa v^{-1}X + (1 - \kappa v^{-1})Z^{-1}, \quad (54)$$

$$Y^\Downarrow = v\kappa^{-1}Y + (1 - v\kappa^{-1})Z^{-1}. \quad (55)$$

Proof: To verify (54) evaluate X, Z, X^\Downarrow using (38), (42) and (52), respectively, and simplify the result. To verify (55) evaluate Y, Z, Y^\Downarrow using (40), (42) and (53), respectively, and simplify the result. \square

We summarize this section with the following theorems.

Theorem 13.17. *Let Φ denote a Leonard system on V as in Definition 3.1. Assume Φ has dual q -Krawtchouk type. Let X, Y, Z denote the corresponding elements from Definition 13.3. Then the following (i)–(iii) hold.*

- (i) *There exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on V such that the equitable generators x, y, z of $U_q(\mathfrak{sl}_2)$ act on V as the maps X, Y, Z , respectively.*
- (ii) *The $U_q(\mathfrak{sl}_2)$ -module V is isomorphic to $L(d, \epsilon)$.*
- (iii) *The inversion of a normalized Φ -split basis for V coincides with a normalized y -eigenbasis for the $U_q(\mathfrak{sl}_2)$ -module V .*

Proof: Combine Lemma 13.4 and Lemma 13.9. \square

Theorem 13.18. *Let Φ denote a Leonard system on V as in Definition 3.1. Assume Φ has dual q -Krawtchouk type. Let $X^\Downarrow, Y^\Downarrow$ denote the corresponding elements from Definition 13.5 and let Z denote the corresponding element from Definition 13.3. Then the following (i)–(iii) hold.*

- (i) *There exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on V such that the equitable generators x, y, z of $U_q(\mathfrak{sl}_2)$ act on V as the maps $X^\Downarrow, Y^\Downarrow, Z$, respectively.*
- (ii) *The $U_q(\mathfrak{sl}_2)$ -module V is isomorphic to $L(d, \epsilon)$.*
- (iii) *The inversion of a normalized Φ^\Downarrow -split basis for V coincides with a normalized y -eigenbasis for the $U_q(\mathfrak{sl}_2)$ -module V .*

Proof: Combine Lemma 13.6 and Lemma 13.10. \square

Theorem 13.19. *Let Φ denote a Leonard system on V as in Definition 3.1. Assume Φ has dual q -Krawtchouk type. Let $h, h^*, \kappa, \kappa^*, v$ denote the corresponding parameters from Definition 11.1. Then there exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on V such that on V ,*

$$\begin{aligned} A &= h1 + \epsilon\kappa x + \epsilon v y, \\ A^* &= h^*1 + \epsilon\kappa^* z, \end{aligned}$$

where x, y, z are the equitable generators for $U_q(\mathfrak{sl}_2)$. This $U_q(\mathfrak{sl}_2)$ -module structure coincides with the $U_q(\mathfrak{sl}_2)$ -module structure from Theorem 13.17.

Proof: The existence follows from Lemma 13.4 and Lemma 13.11. The uniqueness follows from Lemma 13.14 and since $x, y, z^{\pm 1}$ generate $U_q(\mathfrak{sl}_2)$. \square

Theorem 13.20. *Let Φ denote a Leonard system on V as in Definition 3.1. Assume Φ has dual q -Krawtchouk type. Let $h, h^*, \kappa, \kappa^*, v$ denote the corresponding parameters from Definition 11.1. Then there exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on V such that on V ,*

$$\begin{aligned} A &= h1 + \epsilon\kappa y + \epsilon v x, \\ A^* &= h^*1 + \epsilon\kappa^* z, \end{aligned}$$

where x, y, z are the equitable generators for $U_q(\mathfrak{sl}_2)$. This $U_q(\mathfrak{sl}_2)$ -module structure coincides with the $U_q(\mathfrak{sl}_2)$ -module structure from Theorem 13.18.

Proof: Recall that the Leonard system Φ^\downarrow has dual q -Krawtchouk type. Apply Theorem 13.19 to Φ^\downarrow and use Note 11.8 to obtain the result. \square

14 Distance-regular graphs; preliminaries

We now turn our attention to distance-regular graphs. After a brief review of the basic definitions we recall the subconstituent algebra and the Q -polynomial structure. For more information we refer the reader to [2, 4, 10, 22].

Let X denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the \mathbb{C} -algebra consisting of the matrices with entries in \mathbb{C} , and rows and columns indexed by X . Let $V = \mathbb{C}^X$ denote the vector space over \mathbb{C} consisting of the column vectors with entries in \mathbb{C} and rows indexed by X . Observe that $\text{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module* of $\text{Mat}_X(\mathbb{C})$. We endow V with the Hermitean inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v \rangle = u^t \bar{v}$ for $u, v \in V$, where t denotes transpose and $\bar{\cdot}$ denotes complex conjugation. For all $y \in X$, let \hat{y} denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. Observe that $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V .

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X , edge set R , path-length distance function ∂ , and diameter $D := \max\{\partial(x, y) \mid x, y \in X\}$. For $x \in X$ and an integer i ($0 \leq i \leq D$) let $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$ we say Γ is *regular with valency k* whenever $|\Gamma(x)| = k$ for every $x \in X$. We say Γ is *distance-regular* whenever for all integers h, i, j ($0 \leq h, i, j \leq D$) and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{ij}^h = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x and y . The constants p_{ij}^h are called the *intersection numbers* of Γ . We abbreviate $c_i = p_{1, i-1}^i$ ($1 \leq i \leq D$), $b_i = p_{i, i+1}^i$ ($0 \leq i \leq D-1$), $a_i = p_{1i}^i$ ($0 \leq i \leq D$), and $c_0 = 0$, $b_D = 0$.

For the rest of the paper assume Γ is distance-regular with $D \geq 3$. Observe that Γ is regular with valency $k = b_0$. Moreover $k = c_i + a_i + b_i$ for $0 \leq i \leq D$. By the triangle inequality,

for $0 \leq h, i, j \leq D$ we have $p_{ij}^h = 0$ (resp. $p_{ij}^h \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. In particular $c_i \neq 0$ for $1 \leq i \leq D$ and $b_i \neq 0$ for $0 \leq i \leq D-1$.

We recall the Bose-Mesner algebra of Γ . For $0 \leq i \leq D$ let A_i denote the matrix in $\text{Mat}_X(\mathbb{C})$ with (x, y) -entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (x, y \in X).$$

We call A_i the i th *distance matrix* of Γ . We abbreviate $A = A_1$ and call this the *adjacency matrix* of Γ . Observe that (ai) $A_0 = I$; (aia) $J = \sum_{i=0}^D A_i$; (aiia) $\overline{A_i} = A_i$ ($0 \leq i \leq D$); (aiv) $A_i^t = A_i$ ($0 \leq i \leq D$); (av) $A_i A_j = \sum_{h=0}^D p_{ij}^h A_h$ ($0 \leq i, j \leq D$), where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\text{Mat}_X(\mathbb{C})$. Using these facts we find $\{A_i\}_{i=0}^D$ is a basis for a commutative subalgebra M of $\text{Mat}_X(\mathbb{C})$. We call M the *Bose-Mesner algebra* of Γ . By [2, p. 190] A generates M . By [4, p. 45] M has a second basis $\{E_i\}_{i=0}^D$ such that (ei) $E_0 = |X|^{-1}J$; (eii) $I = \sum_{i=0}^D E_i$; (eiii) $\overline{E_i} = E_i$ ($0 \leq i \leq D$); (eiv) $E_i^t = E_i$ ($0 \leq i \leq D$); (ev) $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq D$). We call $\{E_i\}_{i=0}^D$ the *primitive idempotents* of Γ .

We recall the eigenvalues of Γ . Since $\{E_i\}_{i=0}^D$ form a basis for M , there exist complex scalars $\{\theta_i\}_{i=0}^D$ such that $A = \sum_{i=0}^D \theta_i E_i$. Combining this with (ev) we find $AE_i = E_i A = \theta_i E_i$ for $0 \leq i \leq D$. We call θ_i the *eigenvalue* of Γ associated with E_i . By [2, p. 197] the scalars $\{\theta_i\}_{i=0}^D$ are in \mathbb{R} . The $\{\theta_i\}_{i=0}^D$ are mutually distinct since A generates M . By (ei) we have $\theta_0 = k$. By (eii)–(ev),

$$V = E_0 V + E_1 V + \cdots + E_D V \quad (\text{orthogonal direct sum}).$$

For $0 \leq i \leq D$ the space $E_i V$ is the eigenspace of A associated with θ_i . Let m_i denote the rank of E_i and note that m_i is the dimension of $E_i V$. We call m_i the *multiplicity* of E_i (or θ_i).

We recall the Krein parameters of Γ . Let \circ denote the entrywise product in $\text{Mat}_X(\mathbb{C})$. Observe that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \leq i, j \leq D$, so M is closed under \circ . Thus there exist complex scalars q_{ij}^h ($0 \leq h, i, j \leq D$) such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h \quad (0 \leq i, j \leq D).$$

By [3, p. 170], q_{ij}^h is real and nonnegative for $0 \leq h, i, j \leq D$. The q_{ij}^h are called the *Krein parameters* of Γ .

We recall the dual Bose-Mesner algebra of Γ . For the rest of this section we fix a vertex $x \in X$. We view x as a “base vertex”. For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with (y, y) -entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X). \quad (56)$$

We call E_i^* the i th *dual idempotent* of Γ with respect to x [22, p. 378]. Observe that (esi) $I = \sum_{i=0}^D E_i^*$; (esii) $\overline{E_i^*} = E_i^*$ ($0 \leq i \leq D$); (esiii) $E_i^{*t} = E_i^*$ ($0 \leq i \leq D$); (esiv) $E_i^* E_j^* = \delta_{ij} E_i^*$ ($0 \leq i, j \leq D$). By these facts $\{E_i^*\}_{i=0}^D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\text{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose-Mesner algebra* of Γ with respect to x [22, p. 378]. For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with (y, y) -entry $(A_i^*)_{yy} = |X|(\overline{E_i^*})_{xy}$ for $y \in X$. Then $\{A_i^*\}_{i=0}^D$ is a basis for M^* [22, p. 379]. Moreover (asi) $A_0^* = I$; (asii) $\overline{A_i^*} = A_i^*$ ($0 \leq i \leq D$); (asiii) $A_i^{*t} = A_i^*$ ($0 \leq i \leq D$); (asiv) $A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^*$ ($0 \leq i, j \leq D$) [22, p. 379]. We call $\{A_i^*\}_{i=0}^D$ the *dual distance matrices* of Γ with respect to x .

We recall the subconstituents of Γ . From (56) we find

$$E_i^* V = \text{span}\{\hat{y} | y \in X, \partial(x, y) = i\} \quad (0 \leq i \leq D). \quad (57)$$

We call $E_i^* V$ the i th *subconstituent* of Γ with respect to x . By (57) and since $\{\hat{y} | y \in X\}$ is an orthonormal basis for V we find

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \quad (\text{orthogonal direct sum}).$$

We recall the subconstituent algebra of Γ . Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by M and M^* . We call T the *subconstituent algebra* (or *Terwilliger algebra*) of Γ with respect to x [22, Definition 3.3]. Observe that T has finite dimension. Moreover T is semisimple since it is closed under the conjugate transpose map [8, p. 157]. By [22, Lemma 3.2] the following are relations in T .

$$E_h^* A_i E_j^* = 0 \quad \text{iff} \quad p_{ij}^h = 0, \quad (0 \leq h, i, j \leq D), \quad (58)$$

$$E_h A_i^* E_j = 0 \quad \text{iff} \quad q_{ij}^h = 0, \quad (0 \leq h, i, j \leq D). \quad (59)$$

We recall the Q -polynomial property. The graph Γ is said to be *Q -polynomial* (with respect to the given ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents) whenever for $0 \leq h, i, j \leq D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two [4, p. 235].

For the rest of this section assume Γ is Q -polynomial with respect to $\{E_i\}_{i=0}^D$. We abbreviate $c_i^* = q_{1,i-1}^i$ ($1 \leq i \leq D$), $b_i^* = q_{1,i+1}^i$ ($0 \leq i \leq D-1$), $a_i^* = q_{1i}^i$ ($0 \leq i \leq D$), and $c_0^* = 0$, $b_D^* = 0$. We call the sequence $\{\theta_i\}_{i=0}^D$ the *eigenvalue sequence* for this Q -polynomial structure. We abbreviate $A^* = A_1^*$ and call this the *dual adjacency matrix* of Γ with respect to x . The matrix A^* generates M^* [22, Lemma 3.11]. Therefore A, A^* together generate T .

By (58), (59) we have

$$\begin{aligned} E_i^* A E_j^* &= 0 \quad \text{if} \quad |i - j| > 1, \\ E_i A^* E_j &= 0 \quad \text{if} \quad |i - j| > 1 \end{aligned} \quad (60)$$

for $0 \leq i, j \leq D$.

We recall the dual eigenvalues of Γ . Since $\{E_i^*\}_{i=0}^D$ form a basis for M^* there exist complex scalars $\{\theta_i^*\}_{i=0}^D$ such that $A^* = \sum_{i=0}^D \theta_i^* E_i^*$. Combining this with (esiv) we find $A^* E_i^* =$

$E_i^* A^* = \theta_i^* E_i^*$ for $0 \leq i \leq D$. By [22, Lemma 3.11] the scalars $\{\theta_i^*\}_{i=0}^D$ are in \mathbb{R} . The scalars $\{\theta_i^*\}_{i=0}^D$ are mutually distinct since A^* generates M^* . We call θ_i^* the *dual eigenvalue* of Γ associated with E_i^* ($0 \leq i \leq D$). We call the sequence $\{\theta_i^*\}_{i=0}^D$ the *dual eigenvalue sequence* for the given Q -polynomial structure. By [4, Lemma 2.21(ii)], $\theta_0^* = m_1$. For $0 \leq i \leq D$ the space $E_i^* V$ is the eigenspace of A^* associated with θ_i^* .

Recall the tridiagonal relations from Lemma 10.1.

Lemma 14.1. [24, Lemma 5.4] *There exist unique scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ in \mathbb{C} such that both*

$$\begin{aligned} [A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(A A^* + A^* A) - \varrho A^*] &= 0, \\ [A^*, A^{*2} A - \beta A^* A A^* + A A^{*2} - \gamma^*(A^* A + A A^*) - \varrho^* A] &= 0. \end{aligned} \quad (61)$$

Lemma 14.2. [24, Lemma 5.4] *The scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ from Lemma 14.1 satisfy the following (i)–(v).*

(i) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are both equal to $\beta + 1$ for $2 \leq i \leq D - 1$.

(ii) $\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1} \quad (1 \leq i \leq D - 1).$

(iii) $\gamma^* = \theta_{i-1}^* - \beta\theta_i^* + \theta_{i+1}^* \quad (1 \leq i \leq D - 1).$

(iv) $\varrho = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) \quad (1 \leq i \leq D).$

(v) $\varrho^* = \theta_{i-1}^{*2} - \beta\theta_{i-1}^*\theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) \quad (1 \leq i \leq D).$

We recall the T -modules. By a T -module we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. Let W denote a T -module and let W' denote a T -module contained in W . Then the orthogonal complement of W' in W is a T -module [9, p. 802]. It follows that each T -module is an orthogonal direct sum of irreducible T -modules. In particular V is an orthogonal direct sum of irreducible T -modules.

Let W denote an irreducible T -module. Observe that W is the direct sum of the nonzero spaces among $E_0^* W, \dots, E_D^* W$. Similarly W is the direct sum of the nonzero spaces among $E_0 W, \dots, E_D W$. By the *endpoint* of W we mean $\min\{i | 0 \leq i \leq D, E_i^* W \neq 0\}$. By the *diameter* of W we mean $|\{i | 0 \leq i \leq D, E_i^* W \neq 0\}| - 1$. By the *dual endpoint* of W we mean $\min\{i | 0 \leq i \leq D, E_i W \neq 0\}$. By the *dual diameter* of W we mean $|\{i | 0 \leq i \leq D, E_i W \neq 0\}| - 1$. It turns out that the diameter of W is equal to the dual diameter of W [19, Corollary 3.3]. By [22, Lemma 3.9, Lemma 3.12] $\dim E_i^* W \leq 1$ for $0 \leq i \leq D$ if and only if $\dim E_i W \leq 1$ for $0 \leq i \leq D$; in this case W is called *thin*. Γ is called *thin* (with respect to x) whenever all of its irreducible T -modules are thin.

Lemma 14.3. [22, Lemma 3.4, Lemma 3.9, Lemma 3.12] *Let W denote an irreducible T -module. Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. Then r, t, d are nonnegative integers such that $r + d \leq D$ and $t + d \leq D$. Moreover the following (i)–(iv) hold.*

- (i) $E_i^*W \neq 0$ if and only if $r \leq i \leq r+d$, $(0 \leq i \leq D)$.
- (ii) $W = \sum_{h=0}^d E_{r+h}^*W$ (orthogonal direct sum).
- (iii) $E_iW \neq 0$ if and only if $t \leq i \leq t+d$, $(0 \leq i \leq D)$.
- (iv) $W = \sum_{h=0}^d E_{t+h}W$ (orthogonal direct sum).

Lemma 14.4. [7, Lemma 4.1, Lemma 8.7] Let W denote a thin irreducible T -module. Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. For any nonzero $u \in E_tW$, the sequence $\{E_{r+i}^*u\}_{i=0}^d$ is a basis for W . Consider the matrices in $\text{Mat}_{d+1}(\mathbb{C})$ that represent A and A^* with respect to this basis. The matrix representing A is irreducible tridiagonal with constant row sum θ_t , and the matrix representing A^* is $\text{diag}(\theta_r^*, \theta_{r+1}^*, \dots, \theta_{r+d}^*)$.

Definition 14.5. [7, Definition 8.2] Let W denote a thin irreducible T -module. Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. By a *standard basis* for W we mean a sequence $\{E_{r+i}^*u\}_{i=0}^d$ where u is a nonzero vector in E_tW . The matrix in $\text{Mat}_{d+1}(\mathbb{C})$ that represents A with respect to a standard basis will be denoted

$$\begin{pmatrix} a_0(W) & b_0(W) & & & & & & \mathbf{0} \\ c_1(W) & a_1(W) & b_1(W) & & & & & \\ & c_2(W) & \cdot & \cdot & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & b_{d-1}(W) & \\ \mathbf{0} & & & & & c_d(W) & a_d(W) & \end{pmatrix}. \quad (62)$$

We call the matrix (62) the *intersection matrix* of W .

Lemma 14.6. [7, Lemma 4.2, Lemma 8.8] Let W denote a thin irreducible T -module. Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. For any nonzero $v \in E_r^*W$, the sequence $\{E_{t+i}v\}_{i=0}^d$ is a basis for W . Consider the matrices in $\text{Mat}_{d+1}(\mathbb{C})$ that represent A and A^* with respect to this basis. The matrix representing A^* is irreducible tridiagonal with constant row sum θ_r^* , and the matrix representing A is $\text{diag}(\theta_t, \theta_{t+1}, \dots, \theta_{t+d})$.

Definition 14.7. [7, Definition 8.2] Let W denote a thin irreducible T -module. Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. By a *dual standard basis* for W we mean a sequence $\{E_{t+i}v\}_{i=0}^d$ where v is a nonzero vector in E_r^*W . The matrix in $\text{Mat}_{d+1}(\mathbb{C})$ that represents A^* with respect to a dual standard basis will be denoted

$$\begin{pmatrix} a_0^*(W) & b_0^*(W) & & & & & & \mathbf{0} \\ c_1^*(W) & a_1^*(W) & b_1^*(W) & & & & & \\ & c_2^*(W) & \cdot & \cdot & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & \cdot & \cdot & b_{d-1}^*(W) & \\ \mathbf{0} & & & & & c_d^*(W) & a_d^*(W) & \end{pmatrix}. \quad (63)$$

We call the matrix (63) the *dual intersection matrix* of W .

Lemma 14.8. *Let W denote a thin irreducible T -module. Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. Then the sequence*

$$\Phi = (A|_W; \{E_{t+i}|_W\}_{i=0}^d; A^*|_W; \{E_{r+i}^*|_W\}_{i=0}^d)$$

is a Leonard system on W . The intersection matrix of Φ from Definition 9.1 coincides with the intersection matrix of W from (62). The dual intersection matrix of Φ from Definition 9.2 coincides with the dual intersection matrix of W from (63).

Proof: The first assertion follows from Lemma 14.4 and Lemma 14.6. The intersection matrix of Φ coincides with (62) because a Φ -standard basis for W is a standard basis for the T -module W . The dual intersection matrix of Φ coincides with (63) because a Φ^* -standard basis for W is a dual standard basis for the T -module W . \square

Lemma 14.9. *Let W denote a thin irreducible T -module. Then there exist scalars $\omega = \omega(W), \eta = \eta(W), \eta^* = \eta^*(W)$ in \mathbb{C} such that both*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^*A^2 + \omega A + \eta I, \quad (64)$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^*A = \gamma A^{*2} + \omega A^* + \eta^*I \quad (65)$$

on W . In the above equations, the scalars $\beta, \gamma, \gamma^, \varrho, \varrho^*$ are from Lemma 14.1.*

Proof: Apply Lemma 10.3 to the Leonard system from Lemma 14.8. \square

For notational convenience let θ_{-1} and θ_{D+1} (resp. θ_{-1}^* and θ_{D+1}^*) denote the scalars in \mathbb{K} which satisfy Lemma 14.2(ii) (resp. Lemma 14.2(iii)) for $i = 0$ and $i = D$.

Lemma 14.10. *Let W denote a thin irreducible T -module. Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. Let ω, η, η^* denote scalars in \mathbb{C} . Then ω, η, η^* satisfy (64), (65) if and only if the following (i)–(iv) hold.*

$$(i) \quad \omega = a_i(W)(\theta_{r+i}^* - \theta_{r+i+1}^*) + a_{i-1}(W)(\theta_{r+i-1}^* - \theta_{r+i-2}^*) - \gamma(\theta_{r+i}^* + \theta_{r+i-1}^*) \quad (1 \leq i \leq d).$$

$$(ii) \quad \omega = a_i^*(W)(\theta_{t+i} - \theta_{t+i+1}) + a_{i-1}^*(W)(\theta_{t+i-1} - \theta_{t+i-2}) - \gamma^*(\theta_{t+i} + \theta_{t+i-1}) \quad (1 \leq i \leq d).$$

$$(iii) \quad \eta = a_i^*(W)(\theta_{t+i} - \theta_{t+i-1})(\theta_{t+i} - \theta_{t+i+1}) - \gamma^*\theta_{t+i}^2 - \omega\theta_{t+i} \quad (0 \leq i \leq d).$$

$$(iv) \quad \eta^* = a_i(W)(\theta_{r+i}^* - \theta_{r+i-1}^*)(\theta_{r+i}^* - \theta_{r+i+1}^*) - \gamma\theta_{r+i}^{*2} - \omega\theta_{r+i}^* \quad (0 \leq i \leq d).$$

Proof: Apply Lemma 10.4 to the Leonard system from Lemma 14.8. \square

We will be discussing the center of T , denoted $Z(T)$.

Lemma 14.11. *Suppose Γ is thin. Then there exist elements Ω, G, G^* of $Z(T)$ with the following property. For every irreducible T -module W , the elements Ω, G, G^* act on W as $\omega I, \eta I, \eta^* I$ where $\omega = \omega(W), \eta = \eta(W), \eta^* = \eta^*(W)$ are from Lemma 14.9.*

Proof: Let $\{W_i\}_{i=1}^n$ denote a full set of mutually nonisomorphic irreducible T -modules. For $1 \leq i \leq n$ let V_i denote the subspace of V spanned by the irreducible T -modules that are isomorphic to W_i . Observe that each V_i is a T -submodule of V . By construction

$$V = V_1 + V_2 + \cdots + V_n \quad (\text{direct sum}).$$

There exists matrices Ω, G, G^* in $\text{Mat}_X(\mathbb{C})$ that act on V_i as $\omega(W_i)I, \eta(W_i)I, \eta^*(W_i)I$, respectively for $1 \leq i \leq n$. Now consider our irreducible T -module W . By construction there exists an integer j such that W is contained in V_j . Therefore Ω, G, G^* act on W as $\omega(W_j)I, \eta(W_j)I, \eta^*(W_j)I$, respectively. In particular Ω, G, G^* act on W as scalar multiples of I and leave W invariant. By this and a similar argument to the proof of [13, Lemma 12.1], each of Ω, G, G^* is in $Z(T)$. Since the T -module W is irreducible and is contained in V_j , the T -modules W, W_j are isomorphic. Therefore $\omega(W) = \omega(W_j), \eta(W) = \eta(W_j), \eta^*(W) = \eta^*(W_j)$. By these comments the Ω, G, G^* act on W as $\omega(W)I, \eta(W)I, \eta^*(W)I$, respectively. \square

Lemma 14.12. *Suppose Γ is thin. Let $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ denote the scalars from Lemma 14.1. Let Ω, G, G^* denote the elements of T from Lemma 14.11. Then both*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^*A^2 + \Omega A + G, \quad (66)$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \varrho^*A = \gamma A^{*2} + \Omega A^* + G^*. \quad (67)$$

Proof: Write V as a direct sum of irreducible T -modules. By Lemma 14.9 and Lemma 14.11, lines (66), (67) hold on each irreducible T -module in the sum. Therefore lines (66), (67) hold on V . \square

We mention a result about $Z(T)$ for later use.

Lemma 14.13. *Let C denote an element in $Z(T)$. Then for vertices $y, z \in X$*

$$C_{yz} \neq 0 \quad \rightarrow \quad \partial(x, y) = \partial(x, z).$$

Proof: Assume $C_{yz} \neq 0$. Let $i = \partial(x, y)$ and $j = \partial(x, z)$. We show $i = j$. Suppose $i \neq j$. Since $E_i^*C = CE_i^*$ and $E_i^*E_j^* = 0$, we have $E_i^*CE_j^* = E_i^*E_j^*C = 0$. However $(E_i^*CE_j^*)_{yz} = (E_i^*)_{yy}C_{yz}(E_j^*)_{zz} = C_{yz} \neq 0$, a contradiction. Therefore $i = j$. \square

15 Near polygons

We continue to discuss the distance-regular graph Γ from Section 14. In this section we consider the case in which Γ is a near polygon. A clique in Γ is called *maximal* whenever it is not properly contained in another clique.

Definition 15.1. [4, p. 198] The graph Γ is called a *near polygon* whenever the following two axioms hold.

(NP1) There are no induced subgraphs of shape $K_{1,2,1}$.

(NP2) For a vertex x in X and a maximal clique M of Γ with $\partial(x, M) < D$, there exists a unique vertex in M nearest to x .

Definition 15.2. [4, p. 198] Suppose Γ is a near polygon. Then Γ is called a *near n -gon*, where $n = 2D + 1$ if there is a vertex at distance D from some maximal clique, and $n = 2D$ otherwise.

Lemma 15.3. [4, Theorem 6.4.1] *The graph Γ is a near polygon if and only if the axiom (NP1) holds and $a_i = a_1 c_i$ for $1 \leq i \leq D - 1$. In this case Γ is a near $(2D + 1)$ -gon if $a_D \neq a_1 c_D$ and a near $2D$ -gon if $a_D = a_1 c_D$.*

Lemma 15.4. [4, p. 200] *Assume Γ is a near polygon. Then each edge in Γ is contained in a unique maximal clique, and this clique has cardinality $a_1 + 2$.*

Lemma 15.5. *Assume Γ is a near polygon. Fix $x \in X$. Fix adjacent $y, z \in X$ such that $\partial(x, y) = \partial(x, z) - 1$. Define $i = \partial(x, y)$. Then*

$$\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z) = \emptyset, \quad |\Gamma_{i+1}(x) \cap \Gamma(y) \cap \Gamma(z)| = a_1. \quad (68)$$

Proof: Since y, z are adjacent, we have $|\Gamma(y) \cap \Gamma(z)| = a_1$. Since $\partial(x, y) = i$ and $\partial(x, z) = i + 1$, we have $\Gamma(y) \cap \Gamma(z) \subseteq \Gamma_i(x) \cup \Gamma_{i+1}(x)$. By these comments

$$|\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)| + |\Gamma_{i+1}(x) \cap \Gamma(y) \cap \Gamma(z)| = |\Gamma(y) \cap \Gamma(z)| = a_1.$$

By this it suffices to show that $\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z) = \emptyset$. Suppose there exists a vertex $w \in \Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)$. Let C denote a maximal clique of Γ that contains y, z . Observe that $w \in C$ by (NP1). By construction $\partial(x, C) = i$. By (NP2) there exists a unique vertex in C at distance i from x . However y, w are distinct vertices in C at distance i from x , a contradiction. The result follows. \square

We recall some definitions for future use.

Definition 15.6. [31, Definition 3.1] Assume Γ is a near polygon. A subgraph H of Γ is called *weak-geodetically closed* whenever for all $x, y \in H$ and for all $z \in X$

$$\partial(x, z) + \partial(z, y) \leq \partial(x, y) + 1 \quad \rightarrow \quad z \in H.$$

Lemma 15.7. *Assume Γ is a near polygon. Let H denote a weak-geodetically closed subgraph of Γ . For all $x, y \in H$ we have*

$$\partial(x, y) \text{ (in } H) = \partial(x, y) \text{ (in } \Gamma).$$

Proof: Clear. \square

Definition 15.8. [5, p. 145] Assume Γ is a near polygon. A subgraph Q of Γ is called a *quad* whenever Q has diameter 2 and Q is weak geodetically-closed.

16 Dual polar graphs

In this section we discuss a type of near polygon called a dual polar graph. Let b denote a prime power. Let \mathbb{F}_b denote the finite field of order b . Let U denote a finite-dimensional vector space over \mathbb{F}_b endowed with one of the following nondegenerate forms.

name	$\dim U$	form	e
$C_D(b)$	$2D$	symplectic	1
$B_D(b)$	$2D + 1$	quadratic	1
$D_D(b)$	$2D$	quadratic (Witt index D)	0
${}^2D_{D+1}(b)$	$2D + 2$	quadratic (Witt index D)	2
${}^2A_{2D}(q)$	$2D + 1$	Hermitean ($b = q^2$)	$\frac{3}{2}$
${}^2A_{2D-1}(q)$	$2D$	Hermitean ($b = q^2$)	$\frac{1}{2}$

A subspace W of U is called *isotropic* whenever the form vanishes completely on W . By [6, Theorem 6.3.1], each maximal isotropic subspace of U has dimension D . Define a graph as follows. The vertex set consists of the maximal isotropic subspaces of U . Vertices y, z are adjacent whenever $\dim(y \cap z) = D - 1$. By [4, p. 274] this graph is distance-regular and has diameter D . By [4, p. 276] for vertices y, z we have $\partial(y, z) = D - \dim(y \cap z)$. We call this graph the *dual polar graph* associated with U .

For the rest of the paper assume Γ is a dual polar graph. In the next few lemmas we recall some basic data about Γ .

Lemma 16.1. [4, Theorem 9.4.3] *The intersection numbers c_i, b_i of Γ are given by*

$$c_i = \frac{b^i - 1}{b - 1}, \quad b_i = \frac{b^{i+e}(b^{D-i} - 1)}{b - 1} \quad (0 \leq i \leq D).$$

In particular the valency $k = b_0$ is

$$k = \frac{b^e(b^D - 1)}{b - 1}.$$

Corollary 16.2. *The intersection numbers a_i of Γ are given by*

$$a_i = \frac{(b^e - 1)(b^i - 1)}{b - 1} \quad (0 \leq i \leq D).$$

Proof: Use Lemma 16.1 and $a_i = k - c_i - b_i$. □

Lemma 16.3. [6, Proposition 10.4.1] *The graph Γ is a near $2D$ -gon.*

Lemma 16.4. [4, Theorem 9.4.3] *Let $\theta_0 > \theta_1 > \dots > \theta_D$ denote the eigenvalues of Γ . Then*

$$\theta_i = \frac{1 - b^e + b^{D+e-i} - b^i}{b - 1},$$

$$m_i = \frac{b^i(b^D - 1)(b^{D-1} - 1) \dots (b^{D-i+1} - 1)}{(b^i - 1) \dots (b^2 - 1)(b - 1)} \frac{1 + b^{D+e-2i}}{1 + b^{D+e-i}} \prod_{j=1}^i \frac{1 + b^{D+e-j}}{1 + b^{j-e}}$$

for $0 \leq i \leq D$.

Lemma 16.5. *The graph Γ is Q -polynomial with respect to the ordering $\theta_0 > \theta_1 > \dots > \theta_D$ of the eigenvalues. The corresponding dual eigenvalue sequence is*

$$\theta_i^* = \zeta + \xi b^{-i} \quad (0 \leq i \leq D),$$

where

$$\begin{aligned} \zeta &= -\frac{b(b^{D+e-2} + 1)}{b-1}, \\ \xi &= \frac{b^2(b^{D+e-2} + 1)(b^{D+e-1} + 1)}{(b-1)(b^e + b)}. \end{aligned}$$

Proof: By [4, Table 6.1, Corollary 8.4.2] the graph Γ is Q -polynomial with respect to $\{\theta_i\}_{i=0}^D$. By [4, Theorem 8.4.1] the corresponding dual eigenvalues satisfy

$$\frac{\theta_i^*}{\theta_0^*} = 1 + \frac{\theta_1 - k}{k} \frac{b - b^{1-i}}{b-1} \quad (0 \leq i \leq D). \quad (69)$$

In (69) evaluate k using Lemma 16.1 and evaluate θ_0^* using $\theta_0^* = m_1$ and Lemma 16.4. The result follows. \square

From now on it is understood that the eigenvalues of Γ are ordered such that $\theta_0 > \theta_1 > \dots > \theta_D$.

Lemma 16.6. [7, Theorem 17.19] *The dual intersection numbers of Γ are given by*

$$\begin{aligned} c_i^* &= \frac{\xi b^{-i}(b^i - 1)(b^{e-i} + 1)}{(b^{D+e-2i} + 1)(b^{D+e-2i+1} + 1)}, \\ b_i^* &= \frac{\xi(1 - b^{i-D})(b^{-D-e+i} + 1)}{(b^{-D-e+2i} + 1)(b^{-D-e+2i+1} + 1)}, \\ a_i^* &= \theta_0^* - c_i^* - b_i^* \end{aligned}$$

for $0 \leq i \leq D$.

Lemma 16.7. *The scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ from Lemma 14.1 satisfy the following (i)–(v).*

- (i) $\beta = b + b^{-1}$.
- (ii) $\gamma = \frac{(b^e - 1)(b - 1)}{b}$.
- (iii) $\gamma^* = (b - 1)(b^{D+e-2} + 1)$.
- (iv) $\varrho = \frac{(b^e - 1)^2}{b} + b^{D+e-2}(b + 1)^2$.
- (v) $\varrho^* = b(b^{D+e-2} + 1)^2$.

Proof: Routine verification using Lemma 14.2, Lemma 16.4 and Lemma 16.5. \square

17 The structure of an irreducible T -module for a dual polar graph

We continue to discuss the dual polar graph Γ from Section 16. For the rest of the paper fix a vertex $x \in X$ and write $T = T(x)$ for the subconstituent algebra. In this section we recall some basic data about irreducible T -modules.

Lemma 17.1. [24, Example 6.1] *The graph Γ is thin.*

Lemma 17.2. [24, p. 200] *Let W denote an irreducible T -module. Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. Then for $0 \leq i \leq d$*

$$\begin{aligned} c_i(W) &= \frac{b^t(b^i - 1)}{b - 1}, \\ b_i(W) &= \frac{b^{D+e-d-t+i}(b^{d-i} - 1)}{b - 1}, \\ a_i(W) &= \frac{b^{D+e-d-t+i} - b^e - b^{t+i} + 1}{b - 1}. \end{aligned}$$

Lemma 17.3. [7, Theorem 17.17] *Let W denote an irreducible T -module. Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. Then for $0 \leq i \leq d$*

$$\begin{aligned} c_i^*(W) &= \frac{\xi b^{-r-i}(b^i - 1)(b^{D+e-2t-d-i} + 1)}{(b^{D+e-2t-2i} + 1)(b^{D+e-2t-2i+1} + 1)}, \\ b_i^*(W) &= \frac{\xi b^{-r}(1 - b^{i-d})(b^{-D-e+2t+i} + 1)}{(b^{-D-e+2t+2i} + 1)(b^{-D-e+2t+2i+1} + 1)}, \\ a_i^*(W) &= \theta_r^* - b_i^*(W) - c_i^*(W), \end{aligned}$$

where ξ is the scalar from Lemma 16.5.

Lemma 17.4. *Let W denote an irreducible T -module. Let r, t, d denote the endpoint, dual endpoint and diameter of W , respectively. Let ζ, ξ denote the scalars from Lemma 16.5. Let γ, ϱ denote the scalars from Lemma 16.7. Let $\omega = \omega(W), \eta = \eta(W), \eta^* = \eta^*(W)$ denote the scalars in \mathbb{C} that satisfy the following (i)–(iii).*

- (i) $\omega = \xi(b^{-1} - 1)(b^{e+D-d-t-r} - b^{t-r}) - 2\gamma\zeta.$
- (ii) $\eta = \xi b^{-1}(1 - b^e)(b^{e+D-d-r-t} - b^{t-r}) - \xi b^{e+D-d-r-2}(b + 1)(b^{d+1} + 1) - \varrho\zeta.$
- (iii) $\eta^* = -\gamma\zeta^2 - \zeta\omega.$

Then ω, η, η^* satisfy (64), (65).

Proof: We verify Lemma 14.10(i)–(iv) using Lemma 16.4, Lemma 16.5, Lemma 16.7, Lemma 17.2 and Lemma 17.3. The result follows from Lemma 14.10. \square

We give a comment for future use.

Lemma 17.5. *Let W denote an irreducible T -module. Let r, t, d denote the endpoint, dual endpoint and diameter of W , respectively. Then the isomorphism class of W is determined by r, t, d .*

Proof: By Lemma 17.2 the action of A on W is determined by t, d . By Lemma 17.3 the action of A^* on W is determined by r, t, d . By these comments and since A, A^* generate T , the action of T on W is determined by r, t, d . \square

18 Some combinatorial aspects of a dual polar graph

We continue to discuss the dual polar graph Γ from Section 16. In this section we discuss some combinatorial aspects of Γ . Recall the quads of Γ from Definition 15.8.

Lemma 18.1. [6, p. 132] *For all $y, z \in X$ such that $\partial(y, z) = 2$, there exists a unique quad containing y, z .*

Lemma 18.2. [6, p. 132] *Let Q denote a quad in Γ . For all $u \in X$ there exists a unique $v \in Q$ nearest to u . Moreover for any $w \in Q$, we have $\partial(u, w) = \partial(u, v) + \partial(v, w)$.*

Lemma 18.3. *Fix $y, z \in X$ such that $\partial(x, y) = \partial(x, z) - 1$ and $\partial(y, z) = 2$. Define $i = \partial(x, y)$. Then*

$$|\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)| = 1, \quad |\Gamma_{i+1}(x) \cap \Gamma(y) \cap \Gamma(z)| = b. \quad (70)$$

Proof: By Lemma 18.1 there exists a unique quad Q containing y, z . Since $\partial(y, z) = 2$, we have $|\Gamma(y) \cap \Gamma(z)| = c_2$. By Lemma 16.1 we have $c_2 = b + 1$. Since $\partial(x, y) = i$ and $\partial(x, z) = i + 1$, the intersection $\Gamma(y) \cap \Gamma(z) \subseteq \Gamma_i(x) \cup \Gamma_{i+1}(x)$. By these comments

$$|\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)| + |\Gamma_{i+1}(x) \cap \Gamma(y) \cap \Gamma(z)| = |\Gamma(y) \cap \Gamma(z)| = b + 1.$$

Therefore it suffices to show that $|\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)| = 1$. First we show $\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z) \neq \emptyset$. Suppose $\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z) = \emptyset$. Then $\Gamma_{i+1}(x) \cap \Gamma(y) \cap \Gamma(z) \neq \emptyset$. Let $u \in \Gamma_{i+1}(x) \cap \Gamma(y) \cap \Gamma(z)$. Observe that $u \in Q$. Moreover the maximal clique containing the edge uz lies in Q and, by (NP2), contains a unique vertex v at distance i from x . Observe that y, v are distinct vertices in Q each at distance i from x . By these comments, Lemma 18.2 and since Q has diameter 2, there exists a unique vertex p in Q at distance $i - 1$ from x . By Lemma 18.2 the vertices p, y are adjacent. Let P denote a maximal clique of Γ that contains p, y . Observe that P is contained in Q . Since $\partial(y, z) = 2$ and Q has diameter 2, we have $\partial(z, P) = 1$. Thus z is adjacent to a unique vertex in P and by construction it is different from p and y . This vertex is in $\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)$, a contradiction. Hence $\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z) \neq \emptyset$. Now suppose $|\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)| > 1$. Let u, v denote distinct vertices in $\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)$. By (NP1) the vertices u, v are not adjacent. Observe that Q contains u, v . By these comments, Lemma 18.2 and since Q has diameter 2, there exists a unique vertex p in Q at distance $i - 1$ from x . By Lemma 18.2 the vertex p is adjacent to u, v, y . The vertices p, u, v, y induce a subgraph of shape $K_{1,2,1}$ contradicting (NP1). Thus $|\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)| = 1$. The result follows. \square

19 A basis for $Z(T)$

We continue to discuss the dual polar graph Γ from Section 16. Recall the subconstituent algebra T . In this section we display a basis for $Z(T)$. For $0 \leq r, t, d \leq D$, the triple (r, t, d) is called *feasible* whenever there exists an irreducible T -module with endpoint r , dual endpoint t , and diameter d . Let Feas denote the set of all feasible triples. For $\lambda = (r, t, d) \in \text{Feas}$, let V_λ denote the subspace of V spanned by the irreducible T -modules with endpoint r , dual endpoint t and diameter d . Observe that V_λ is a T -module. We call V_λ the *homogeneous component of V associated with λ* . Observe

$$V = \sum_{\lambda \in \text{Feas}} V_\lambda \quad (\text{direct sum}).$$

For all $\lambda \in \text{Feas}$, define the linear transformation $E_\lambda : V \rightarrow V$ by

$$\begin{aligned} (E_\lambda - I)V_\lambda &= 0, \\ E_\lambda V_{\lambda'} &= 0 \quad \text{if } \lambda' \neq \lambda \quad (\lambda' \in \text{Feas}). \end{aligned}$$

Observe that

$$\begin{aligned} I &= \sum_{\lambda \in \text{Feas}} E_\lambda, \\ E_\lambda E_{\lambda'} &= \delta_{\lambda\lambda'} E_\lambda \quad (\lambda, \lambda' \in \text{Feas}). \end{aligned}$$

Lemma 19.1. [8, Theorem 25.15, Theorem 26.4] *The elements $\{E_\lambda | \lambda \in \text{Feas}\}$ form a basis for the vector space $Z(T)$.*

20 The central elements Ω, G^*

We continue to discuss the dual polar graph Γ from Section 16. Recall the subconstituent algebra T and its central elements Ω, G^* from Lemma 14.11. In this section we show that G^* is a linear combination of Ω and I . Then we display Ω in a certain attractive form. Using this form we obtain a characterization of Ω .

Lemma 20.1. *The elements Ω, G^* are related by*

$$G^* = -\gamma\zeta^2 I - \zeta\Omega \tag{71}$$

where ζ is from Lemma 16.5 and γ is from Lemma 16.7.

Proof: By construction V is a direct sum of irreducible T -modules. Let W denote an irreducible T -module in the sum. It suffices to show that the two sides of (71) agree on W . By Lemma 14.11 the elements Ω, G^* act on W as $\omega(W)I, \eta^*(W)I$, respectively. By Lemma 17.4(iii) we have $\eta^*(W) = -\gamma\zeta^2 - \zeta\omega(W)$. Therefore the two sides of (71) agree on W . The result follows. \square

Lemma 20.2. *We have*

$$\Omega = \sum_{i=1}^D \alpha_i E_i^* A E_i^* + \sum_{i=0}^D \beta_i E_i^* \quad (72)$$

where

$$\begin{aligned} \alpha_i &= \frac{(b^{D+e-1} + 1)(b^{D+e-2} + 1)(1-b)b^{-i}}{b^{e-1} + 1} \quad (1 \leq i \leq D), \\ \beta_i &= \frac{(b^{D+e-2} + 1)(b^e - 1)(2b^{e-1} + 2 - (b^{D+e-1} + 1)b^{-i})}{b^{e-1} + 1} \quad (0 \leq i \leq D). \end{aligned}$$

Proof: Let $\tilde{\Omega}$ denote the expression on the right of (72). By construction V is a direct sum of irreducible T -modules. Let W denote an irreducible T -module in the sum. To show $\tilde{\Omega} = \Omega$ it suffices to show that $\tilde{\Omega}, \Omega$ agree on W . Let r, d denote the endpoint and diameter of W . By construction for $r \leq i \leq r + d$ the element $\tilde{\Omega}$ acts on $E_i^* W$ as $(\alpha_i a_{i-r}(W) + \beta_i)I$. By evaluating $a_{i-r}(W)$ using Lemma 17.2 we find $\alpha_i a_{i-r}(W) + \beta_i = \omega(W)$ where $\omega(W)$ is the scalar from Lemma 17.4(i). Therefore $\tilde{\Omega}$ acts on $E_i^* W$ as $\omega(W)I$. By this and since W is a direct sum of $\{E_i^* W\}_{i=r}^{r+d}$, the element $\tilde{\Omega}$ acts on W as $\omega(W)I$. By Lemma 14.11, the element Ω acts on W as $\omega(W)I$. Therefore $\Omega, \tilde{\Omega}$ agree on W . The result follows. \square

Motivated by Lemma 20.2 we consider an element C of T of the form

$$C := \sum_{i=1}^D \alpha_i E_i^* A E_i^* + \sum_{i=0}^D \beta_i E_i^* \quad (73)$$

for some arbitrary scalars $\alpha_i \in \mathbb{C}$ ($1 \leq i \leq D$) and $\beta_i \in \mathbb{C}$ ($0 \leq i \leq D$).

Observe that C is invariant under transposition. We find necessary and sufficient conditions on α_i, β_i for C to be central in T . By construction C commutes with A^* . Since A, A^* generate T , the element C is central in T if and only if C commutes with A .

Lemma 20.3. *For vertices $y, z \in X$, the (y, z) -entry of C is described as follows. First assume $\partial(x, y) \neq \partial(x, z)$. Then the (y, z) -entry of C is zero. Next assume $\partial(x, y) = \partial(x, z)$ and let s denote this common distance. Then the (y, z) -entry of C is given by*

Case	(y, z) -entry of C
$y = z$	β_s
$\partial(y, z) = 1$	α_s
$\partial(y, z) \geq 2$	0

Proof: Routine consequence of (73). \square

Lemma 20.4. *For integers i, j ($0 \leq i, j \leq D$) and for vertices $y, z \in X$, the (y, z) -entries of $E_i^* A E_j^* A$ and $A E_i^* A E_j^*$ are described as follows.*

$$\begin{aligned}
\text{(i)} \quad (E_i^* A E_j^* A)_{yz} &= \begin{cases} 0, & \text{if } \partial(x, y) \neq i; \\ |\Gamma_j(x) \cap \Gamma(y) \cap \Gamma(z)|, & \text{if } \partial(x, y) = i. \end{cases} \\
\text{(ii)} \quad (A E_i^* A E_j^*)_{yz} &= \begin{cases} 0, & \text{if } \partial(x, z) \neq j; \\ |\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)|, & \text{if } \partial(x, z) = j. \end{cases}
\end{aligned}$$

Proof: (i) We have $(E_i^* A E_j^* A)_{yz} = \sum_{w \in X} (E_i^* A E_j^*)_{yw} A_{wz}$. The result follows.

(ii) By (i) above and since $A E_i^* A E_j^* = (E_j^* A E_i^* A)^t$. \square

Lemma 20.5. *Let y, z denote vertices in X such that $\partial(y, z) \geq 3$ or $|\partial(x, y) - \partial(x, z)| \geq 2$. Then the (y, z) -entries of AC and CA are both zero.*

Proof: First we show $(AC)_{yz} = 0$. By (73) it suffices to show that each of $(A E_i^* A E_i^*)_{yz}$ and $(A E_i^*)_{yz}$ is 0 for $0 \leq i \leq D$. Let i be given. By construction $A_{yz} = 0$ so $(A E_i^*)_{yz} = A_{yz}(E_i^*)_{zz} = 0$. By Lemma 20.4(ii), we have $(A E_i^* A E_i^*)_{yz} = 0$. Therefore $(AC)_{yz} = 0$. By swapping the roles of y and z we have $(AC)_{zy} = 0$. By this and since $CA = (AC)^t$, we have $(CA)_{yz} = (AC)_{zy} = 0$. \square

Lemma 20.6. *Let y, z denote vertices in X such that $\partial(x, y) = \partial(x, z) - 1$ and $\partial(y, z) = 2$. Then the following are equivalent:*

- (i) $(AC)_{yz} = (CA)_{yz}$.
- (ii) $b\alpha_{i+1} = \alpha_i$ where $i = \partial(x, y)$.

Proof: Consider the (y, z) -entries of AC and CA .

$$\begin{aligned}
(AC)_{yz} &= \sum_{j=1}^D \alpha_j (A E_j^* A E_j^*)_{yz} + \sum_{j=0}^D \beta_j (A E_j^*)_{yz} \\
&= \alpha_{i+1} |\Gamma_{i+1}(x) \cap \Gamma(y) \cap \Gamma(z)| \quad \text{by Lemma 20.4(ii)} \\
&= b\alpha_{i+1} \quad \text{by the equation on the right of (70),} \\
(CA)_{yz} &= \sum_{j=1}^D \alpha_j (E_j^* A E_j^* A)_{yz} + \sum_{j=0}^D \beta_j (E_j^* A)_{yz} \\
&= \alpha_i |\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)| \quad \text{by Lemma 20.4(i)} \\
&= \alpha_i \quad \text{by the equation on the left of (70).}
\end{aligned}$$

The result follows. \square

Lemma 20.7. *Let y, z denote vertices in X such that $\partial(x, y) = \partial(x, z) - 1$ and $\partial(y, z) = 1$. Then the following are equivalent:*

- (i) $(AC)_{yz} = (CA)_{yz}$.

(ii) $a_1\alpha_{i+1} + \beta_{i+1} = \beta_i$ where $i = \partial(x, y)$.

Proof: Consider the (y, z) -entries of AC and CA .

$$\begin{aligned}
(AC)_{yz} &= \sum_{j=1}^D \alpha_j (AE_j^* AE_j^*)_{yz} + \sum_{j=0}^D \beta_j (AE_j^*)_{yz} \\
&= \alpha_{i+1} |\Gamma_{i+1}(x) \cap \Gamma(y) \cap \Gamma(z)| + \beta_{i+1} \quad \text{by Lemma 20.4(ii)} \\
&= a_1 \alpha_{i+1} + \beta_{i+1} \quad \text{by the equation on the right of (68),} \\
(CA)_{yz} &= \sum_{j=1}^D \alpha_j (E_j^* AE_j^* A)_{yz} + \sum_{j=0}^D \beta_j (E_j^* A)_{yz} \\
&= \alpha_i |\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)| + \beta_i \quad \text{by Lemma 20.4(i)} \\
&= \beta_i \quad \text{by the equation on the left of (68).}
\end{aligned}$$

The result follows. \square

Lemma 20.8. *Let y, z denote vertices in X such that $\partial(x, y) = \partial(x, z)$ and $\partial(y, z) \leq 2$. Then*

$$(AC)_{yz} = (CA)_{yz}.$$

Proof: Let i denote the common value of $\partial(x, y)$ and $\partial(x, z)$. We have

$$\begin{aligned}
(AE_i^* AE_i^*)_{yz} &= (E_i^* AE_i^* AE_i^*)_{yz} = (E_i^* AE_i^* A)_{yz}, \\
(AE_i^*)_{yz} &= (E_i^* AE_i^*)_{yz} = (E_i^* A)_{yz}.
\end{aligned}$$

The result follows from this and (73). \square

By combining Lemma 20.5, Lemma 20.6, Lemma 20.7 and Lemma 20.8 we have the following result.

Theorem 20.9. *Consider the element C from (73). Then C is central in T if and only if the following (i), (ii) hold.*

$$(i) \quad \alpha_i = b^{1-i} \alpha_1 \quad (1 \leq i \leq D).$$

$$(ii) \quad \beta_i = \beta_0 - a_1 b^{1-i} \frac{b^i - 1}{b - 1} \alpha_1 \quad (0 \leq i \leq D).$$

Corollary 20.10. *Let \mathcal{C} denote the subspace of T consisting of central elements of the form (73). Then Ω and I form a basis for \mathcal{C} .*

Proof: In the formulas of α_i, β_i in Theorem 20.9, there are two free variables α_1, β_0 . Therefore $\dim \mathcal{C} \leq 2$. Observe that $I \in \mathcal{C}$. By Lemma 20.2 the element $\Omega \in \mathcal{C}$. Since I, Ω are linearly independent, they form a basis for \mathcal{C} . \square

21 The central element G

We continue to discuss the dual polar graph Γ from Section 16. Recall the subconstituent algebra T . In this section we investigate the central element $G \in T$ from Lemma 14.11.

Lemma 21.1. *We have*

$$\begin{aligned} G = & \xi(1 - b^2) \sum_{i=1}^D b^{-i} E_i^* A E_{i-1}^* A E_i^* + \xi(1 - b^{-2}) \sum_{i=0}^{D-1} b^{-i} E_i^* A E_{i+1}^* A E_i^* \\ & + \xi(b^{-1} - 1)(b^e - 1) \sum_{i=1}^D b^{-i} E_i^* A E_i^* - \varrho A^*, \end{aligned} \quad (74)$$

where ξ is from Lemma 16.5 and ϱ is from Lemma 16.7.

Proof: Recall that $I = \sum_{i=0}^D E_i^*$, $A^* = \sum_{i=0}^D \theta_i^* E_i^*$, and $E_i^* E_j^* = \delta_{ij} E_i^*$ for $0 \leq i, j \leq D$. By these facts and since G is central in T ,

$$G = \left(\sum_{i=0}^D E_i^* \right) G \left(\sum_{j=0}^D E_j^* \right) = \sum_{i=0}^D \sum_{j=0}^D E_i^* G E_j^* = \sum_{i=0}^D E_i^* G E_i^*.$$

For $0 \leq i \leq D$ we compute $E_i^* G E_i^*$. By (66) and since $A^* E_j^* = E_j^* A^* = \theta_j^* E_j^*$ ($0 \leq j \leq D$),

$$\begin{aligned} E_i^* G E_i^* = & (2\theta_i^* - \gamma^*) E_i^* A^2 E_i^* - \beta E_i^* A A^* A E_i^* \\ & - 2\gamma \theta_i^* E_i^* A E_i^* - \varrho \theta_i^* E_i^* - E_i^* \Omega A E_i^*. \end{aligned} \quad (75)$$

We now evaluate the right-hand side of (75). First assume $1 \leq i \leq D - 1$. Using (60),

$$\begin{aligned} E_i^* A^2 E_i^* &= E_i^* A \left(\sum_{j=0}^D E_j^* \right) A E_i^* \\ &= E_i^* A E_{i-1}^* A E_i^* + E_i^* A E_i^* A E_i^* + E_i^* A E_{i+1}^* A E_i^*, \\ E_i^* A A^* A E_i^* &= E_i^* A \left(\sum_{j=0}^D \theta_j^* E_j^* \right) A E_i^* \\ &= \theta_{i-1}^* E_i^* A E_{i-1}^* A E_i^* + \theta_i^* E_i^* A E_i^* A E_i^* + \theta_{i+1}^* E_i^* A E_{i+1}^* A E_i^*. \end{aligned}$$

Using (72),

$$E_i^* \Omega A E_i^* = \alpha_i E_i^* A E_i^* A E_i^* + \beta_i E_i^* A E_i^*$$

where α_i, β_i are from Lemma 20.2. Evaluating the right-hand side of (75) using the above comments,

$$\begin{aligned} E_i^* G E_i^* = & (2\theta_i^* - \gamma^* - \beta\theta_{i-1}^*) E_i^* A E_{i-1}^* A E_i^* + (2\theta_i^* - \gamma^* - \beta\theta_i^* - \alpha_i) E_i^* A E_i^* A E_i^* \\ & + (2\theta_i^* - \gamma^* - \beta\theta_{i+1}^*) E_i^* A E_{i+1}^* A E_i^* - (2\gamma\theta_i^* + \beta_i) E_i^* A E_i^* - \varrho \theta_i^* E_i^*. \end{aligned}$$

By a similar argument,

$$\begin{aligned} E_0^* G E_0^* &= (2\theta_0^* - \gamma^* - \beta\theta_1^*) E_0^* A E_1^* A E_0^* - \varrho\theta_0^* E_0^*, \\ E_D^* G E_D^* &= (2\theta_D^* - \gamma^* - \beta\theta_{D-1}^*) E_D^* A E_{D-1}^* A E_D^* + (2\theta_D^* - \gamma^* - \beta\theta_D^* - \alpha_D) E_D^* A E_D^* A E_D^* \\ &\quad - (2\gamma\theta_D^* + \beta_D) E_D^* A E_D^* - \varrho\theta_D^* E_D^*. \end{aligned}$$

In the preceding equations we now evaluate the coefficients on the right-hand side. The θ_i^* are from Lemma 16.5, the β, γ, γ^* are from Lemma 16.7, and α_i, β_i are from Lemma 20.2. By these lemmas,

$$\begin{aligned} 2\theta_i^* - \gamma^* - \beta\theta_{i-1}^* &= \xi(1 - b^2)b^{-i} \quad (1 \leq i \leq D), \\ 2\theta_i^* - \gamma^* - \beta\theta_i^* - \alpha_i &= 0 \quad (1 \leq i \leq D), \\ 2\theta_i^* - \gamma^* - \beta\theta_{i+1}^* &= \xi(1 - b^{-2})b^{-i} \quad (0 \leq i \leq D-1), \\ 2\gamma\theta_i^* + \beta_i &= \xi(1 - b^{-1})(b^e - 1)b^{-i} \quad (0 \leq i \leq D). \end{aligned}$$

The result follows. \square

Corollary 21.2. *For vertices $y, z \in X$, the (y, z) -entry of G is described as follows. First assume $\partial(x, y) \neq \partial(x, z)$. Then the (y, z) -entry of G is zero. Next assume $\partial(x, y) = \partial(x, z)$ and let s denote this common distance. Then the (y, z) -entry of G is given in the table below.*

Case	(y, z) -entry of G
$y = z$	$\xi b^{-s-1}(1 - b^2)(bc_s - b^{-1}b_s) - \varrho\theta_s^*$
$\partial(y, z) = 1$	$\xi b^{-s-1}(1 - b)(b^2 + b + b^e - 1)$
$\partial(y, z) = 2, \Gamma_{s+1}(x) \cap \Gamma(y) \cap \Gamma(z) = \emptyset$	$\xi b^{-s}(1 - b^2) \Gamma_{s-1}(x) \cap \Gamma(y) \cap \Gamma(z) $
$\partial(y, z) = 2, \Gamma_{s+1}(x) \cap \Gamma(y) \cap \Gamma(z) \neq \emptyset$	$-\xi b^{-s-1}(b + 1)(b - 1)^2$
$\partial(y, z) \geq 3$	0

In the above table ξ is from Lemma 16.5 and ϱ is from Lemma 16.7.

Proof: The first assertion follows from Lemma 14.13. Now suppose $\partial(x, y) = \partial(x, z) = s$. We verify the table. By (74), the (y, z) -entry of G is given by

$$\begin{aligned} G_{yz} &= \xi(1 - b^2)b^{-s}(E_s^* A E_{s-1}^* A E_s^*)_{yz} + \xi(1 - b^{-2})b^{-s}(E_s^* A E_{s+1}^* A E_s^*)_{yz} \\ &\quad + \xi(b^{-1} - 1)(b^e - 1)b^{-s}(E_s^* A E_s^*)_{yz} - \varrho(A^*)_{yz}. \end{aligned}$$

In the above equation the terms on the right-hand side are given by

$$\begin{aligned} (E_s^* A E_{s-1}^* A E_s^*)_{yz} &= |\Gamma_{s-1}(x) \cap \Gamma(y) \cap \Gamma(z)| \quad \text{by Lemma 20.4,} \\ (E_s^* A E_{s+1}^* A E_s^*)_{yz} &= |\Gamma_{s+1}(x) \cap \Gamma(y) \cap \Gamma(z)| \quad \text{by Lemma 20.4,} \\ (E_s^* A E_s^*)_{yz} &= \begin{cases} 0 & \text{if } \partial(y, z) \neq 1; \\ 1 & \text{if } \partial(y, z) = 1, \end{cases} \\ (A^*)_{yz} &= \sum_{i=0}^D \theta_i^*(E_i^*)_{yz} = \theta_s^*(E_s^*)_{yz} = \delta_{yz}\theta_s^*. \end{aligned}$$

We now split our argument into cases.

Case $y = z$: We have $|\Gamma_{s-1}(x) \cap \Gamma(y) \cap \Gamma(z)| = |\Gamma_{s-1}(x) \cap \Gamma(y)| = c_s$ and $|\Gamma_{s+1}(x) \cap \Gamma(y) \cap \Gamma(z)| = |\Gamma_{s+1}(x) \cap \Gamma(y)| = b_s$. By these comments G_{yz} is as shown in the table.

Case $\partial(y, z) = 1$: By (NP1) the vertices y, z together with $\Gamma(y) \cap \Gamma(z)$ form a maximal clique of Γ . By Lemma 16.3 this clique is at distance less than D from x . By (NP2) we have $|\Gamma_{s-1}(x) \cap \Gamma(y) \cap \Gamma(z)| = 1$ and so $|\Gamma_{s+1}(x) \cap \Gamma(y) \cap \Gamma(z)| = 0$. By this and the preliminary comments we find that G_{yz} is as shown in the table.

Case $\partial(y, z) = 2$: Let Q denote a quad containing y, z . Observe that $\Gamma(y) \cap \Gamma(z)$ is contained in Q . First assume $\Gamma_{s+1}(x) \cap \Gamma(y) \cap \Gamma(z) = \emptyset$. By the preliminary comments we find that G_{yz} is as shown in the table. Next assume $\Gamma_{s+1}(x) \cap \Gamma(y) \cap \Gamma(z) \neq \emptyset$. Then $\partial(x, Q) = s - 1$ by Lemma 18.2 and since Q has diameter 2. By Lemma 18.2 we have $|\Gamma_{s-1}(x) \cap \Gamma(y) \cap \Gamma(z)| = 1$. Let u denote the unique vertex in $\Gamma_{s-1}(x) \cap \Gamma(y) \cap \Gamma(z)$. We claim $\Gamma_s(x) \cap \Gamma(y) \cap \Gamma(z) = \emptyset$. Suppose there exists a vertex v in $\Gamma_s(x) \cap \Gamma(y) \cap \Gamma(z)$. Then v is adjacent to u by Lemma 18.2. Now u, v, y, z induce a subgraph of shape $K_{1,2,1}$ which contradicts (NP1). Hence the claim holds. We have $|\Gamma(y) \cap \Gamma(z)| = c_2$ and $c_2 = b + 1$ by Lemma 16.1. By these comments and since $|\Gamma(y) \cap \Gamma(z)| = |\Gamma_{s-1}(x) \cap \Gamma(y) \cap \Gamma(z)| + |\Gamma_s(x) \cap \Gamma(y) \cap \Gamma(z)| + |\Gamma_{s+1}(x) \cap \Gamma(y) \cap \Gamma(z)|$, we have $|\Gamma_{s+1}(x) \cap \Gamma(y) \cap \Gamma(z)| = b$. By this and the preliminary comments we find that G_{yz} is as shown in the table.

Case $\partial(y, z) \geq 3$: We have $\Gamma(y) \cap \Gamma(z) = \emptyset$ so $|\Gamma_{s-1}(x) \cap \Gamma(y) \cap \Gamma(z)| = 0$ and $|\Gamma_{s+1}(x) \cap \Gamma(y) \cap \Gamma(z)| = 0$. By this and the preliminary comments we find that G_{yz} is as shown in the table. \square

22 The central elements Υ, Ψ, Λ

We continue to discuss the dual polar graph Γ from Section 16. Recall q from Section 11 and b from Section 16. For the rest of the paper b, q are related as follows.

$$b = q^2.$$

We note that q is nonzero and not a root of unity.

In Lemma 14.11 we discussed the central elements Ω, G, G^* of the subconstituent algebra T . In this section we introduce three more central elements Υ, Ψ, Λ of T . These central elements will be useful later when we display some $U_q(\mathfrak{sl}_2)$ -module structures on the standard module V .

Definition 22.1. [18, Definition 3.1] Let W denote an irreducible T -module. Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. By the *displacement of W of the first kind* we mean $r + t + d - D$. By the *displacement of W of the second kind* we mean $r - t$.

Lemma 22.2. [18, Lemma 3.2] *Let W denote an irreducible T -module. Then the following (i), (ii) hold.*

- (i) *Let μ denote the displacement of W of the first kind. Then $0 \leq \mu \leq D$.*
- (ii) *Let ν denote the displacement of W of the second kind. Then $-D \leq \nu \leq D$.*

Definition 22.3. For an integer μ ($0 \leq \mu \leq D$) let V_μ denote the subspace of V spanned by the irreducible T -modules for which μ is the displacement of the first kind. Observe that V_μ is a T -module. By [25, Lemma 4.4] we have $V = \sum_{\mu=0}^D V_\mu$ (orthogonal direct sum). For $0 \leq \mu \leq D$ we define a matrix $\sigma_\mu \in \text{Mat}_X(\mathbb{C})$ such that

$$\begin{aligned} (\sigma_\mu - I)V_\mu &= 0, \\ \sigma_\mu V_{\mu'} &= 0 \quad \text{if } \mu \neq \mu' \quad (0 \leq \mu' \leq D). \end{aligned}$$

In other words σ_μ is the projection from V onto V_μ . We note that $V_\mu = \sigma_\mu V$.

The following three lemmas are immediate from Definition 22.3.

Lemma 22.4. *The following (i), (ii) hold.*

- (i) $I = \sum_{\mu=0}^D \sigma_\mu$.
- (ii) $\sigma_\mu \sigma_{\mu'} = \delta_{\mu\mu'} \sigma_\mu \quad (0 \leq \mu, \mu' \leq D)$.

Lemma 22.5. *We have*

$$V = \sum_{\mu=0}^D \sigma_\mu V \quad (\text{orthogonal direct sum}).$$

Moreover for $0 \leq \mu \leq D$ the subspace $\sigma_\mu V$ is spanned by the irreducible T -modules for which μ is the displacement of the first kind.

Recall the set Feas from Section 19.

Lemma 22.6. *For $0 \leq \mu \leq D$ we have*

$$\sigma_\mu V = \sum V_{(r,t,d)}$$

where the sum is over all $(r, t, d) \in \text{Feas}$ such that $r + t + d - D = \mu$.

Definition 22.7. Let Υ denote the matrix in $\text{Mat}_X(\mathbb{C})$ such that

$$\Upsilon = \sum_{\mu=0}^D q^\mu \sigma_\mu.$$

Lemma 22.8. *For $0 \leq \mu \leq D$ the matrix Υ acts on $\sigma_\mu V$ as $q^\mu I$.*

Proof: Immediate from Lemma 22.4(ii). □

Lemma 22.9. *The matrix Υ is invertible and its inverse is*

$$\Upsilon^{-1} = \sum_{\mu=0}^D q^{-\mu} \sigma_\mu.$$

Proof: Immediate from Lemma 22.4. □

Definition 22.10. For an integer ν ($-D \leq \nu \leq D$) let V_ν denote the subspace of V spanned by the irreducible T -modules for which ν is the displacement of the second kind. Observe that V_ν is a T -module. By [25, Lemma 4.4] we have $V = \sum_{\nu=-D}^D V_\nu$ (orthogonal direct sum). For $-D \leq \nu \leq D$ we define a matrix $\psi_\nu \in \text{Mat}_X(\mathbb{C})$ such that

$$\begin{aligned} (\psi_\nu - I)V_\nu &= 0, \\ \psi_\nu V_{\nu'} &= 0 \quad \text{if } \nu \neq \nu' \quad (-D \leq \nu' \leq D). \end{aligned}$$

In other words ψ_ν is the projection from V onto V_ν . We note that $V_\nu = \psi_\nu V$.

The following three lemmas are immediate from Definition 22.10.

Lemma 22.11. *The following (i), (ii) hold.*

- (i) $I = \sum_{\nu=-D}^D \psi_\nu$.
- (ii) $\psi_\nu \psi_{\nu'} = \delta_{\nu\nu'} \psi_\nu \quad (-D \leq \nu, \nu' \leq D)$.

Lemma 22.12. *We have*

$$V = \sum_{\nu=-D}^D \psi_\nu V \quad (\text{orthogonal direct sum}).$$

Moreover for $-D \leq \nu \leq D$ the subspace $\psi_\nu V$ is spanned by the irreducible T -modules for which ν is the displacement of the second kind.

Lemma 22.13. *For $-D \leq \nu \leq D$ we have*

$$\psi_\nu V = \sum V_{(r,t,d)}$$

where the sum is over all $(r, t, d) \in \text{Feas}$ such that $r - t = \nu$.

Definition 22.14. Let Ψ denote the matrix in $\text{Mat}_X(\mathbb{C})$ such that

$$\Psi = \sum_{\nu=-D}^D q^\nu \psi_\nu.$$

Lemma 22.15. *For $-D \leq \nu \leq D$ the matrix Ψ acts on $\psi_\nu V$ as $q^\nu I$.*

Proof: Immediate from Lemma 22.11(ii). □

Lemma 22.16. *The matrix Ψ is invertible and its inverse is*

$$\Psi^{-1} = \sum_{\nu=-D}^D q^{-\nu} \psi_\nu.$$

Proof: Immediate from Lemma 22.11. □

Earlier we defined the notion of diameter for an irreducible T -module. Using this notion we define some more projections involving V .

Definition 22.17. For an integer d ($0 \leq d \leq D$) let V_d denote the subspace of V spanned by the irreducible T -modules of diameter d . Observe that V_d is a T -module, and $V = \sum_{d=0}^D V_d$ (orthogonal direct sum). For $0 \leq d \leq D$ we define a matrix $\rho_d \in \text{Mat}_X(\mathbb{C})$ such that

$$\begin{aligned} (\rho_d - I)V_d &= 0, \\ \rho_d V_{d'} &= 0 \quad \text{if } d \neq d' \quad (0 \leq d' \leq D). \end{aligned}$$

In other words ρ_d is the projection from V onto V_d . We note that $V_d = \rho_d V$.

The following three lemmas are immediate from Definition 22.17.

Lemma 22.18. *The following (i), (ii) hold.*

- (i) $I = \sum_{d=0}^D \rho_d$.
- (ii) $\rho_d \rho_{d'} = \delta_{dd'} \rho_d \quad (0 \leq d, d' \leq D)$.

Lemma 22.19. *We have*

$$V = \sum_{d=0}^D \rho_d V \quad (\text{orthogonal direct sum}). \tag{76}$$

Moreover for $0 \leq d \leq D$ the subspace $\rho_d V$ is spanned by the irreducible T -modules of diameter d .

Lemma 22.20. *For $0 \leq d \leq D$ we have*

$$\rho_d V = \sum V_{(r,t,d)}$$

where the sum is over all integers r, t such that $(r, t, d) \in \text{Feas}$.

Definition 22.21. Let Λ denote the matrix in $\text{Mat}_X(\mathbb{C})$ such that

$$\Lambda = \sum_{d=0}^D \frac{q^{d+1} + q^{-d-1}}{(q - q^{-1})^2} \rho_d.$$

Lemma 22.22. *For $0 \leq d \leq D$ the matrix Λ acts on $\rho_d V$ as*

$$\frac{q^{d+1} + q^{-d-1}}{(q - q^{-1})^2} I. \tag{77}$$

Proof: Immediate from Lemma 22.18(ii). □

Lemma 22.23. [13, Lemma 12.1] *The matrices Υ, Ψ, Λ are central elements in T .*

The central elements Ω, G, G^* of T from Lemma 14.11 are related to Υ, Ψ, Λ as follows.

Proposition 22.24. *Let ξ, ζ denote the scalars from Lemma 16.5, and let γ, ϱ denote the scalars from Lemma 16.7. Then Ω, G, G^* can be expressed in terms of Υ, Ψ, Λ as follows.*

- (i) $\Omega = \xi(q^{-2} - 1)(q^{2e}\Upsilon^{-2} - \Psi^{-2}) - 2\gamma\zeta I.$
- (ii) $G = \xi q^{-2}(1 - q^{2e})(q^{2e}\Upsilon^{-2} - \Psi^{-2}) - \xi q^{D+2e-5}(q^2 - 1)^2(q^2 + 1)\Upsilon^{-1}\Psi^{-1}\Lambda - \varrho\zeta I.$
- (iii) $G^* = \zeta\xi(1 - q^{-2})(q^{2e}\Upsilon^{-2} - \Psi^{-2}) + \gamma\zeta^2 I.$

Proof: (i) By construction V is a direct sum of irreducible T -modules. Let W denote an irreducible T -module in the sum. It suffices to show that for the equation in (i) the two sides agree on W . Let μ, ν denote the displacement of W of the first kind and second kind, respectively. By Lemma 14.11 the element Ω acts on W as $\omega(W)I$. By Lemma 22.5 we have $W \subseteq \sigma_\mu V$. By this and Lemma 22.8 the matrix Υ acts on W as $q^\mu I$. By Lemma 22.12 we have $W \subseteq \psi_\nu V$. By this and Lemma 22.15 the matrix Ψ acts on W as $q^\nu I$. By these comments, Lemma 17.4(i) and Definition 22.1, the two sides agree on W . The result follows.
(ii) Similar to the proof of (i).
(iii) Combine Lemma 20.1 and (i). □

23 Irreducible T -modules and Leonard systems of dual q -Krawtchouk type

We continue to discuss the dual polar graph Γ from Section 16. In Lemma 14.8 we obtained a Leonard system on each irreducible T -module. In this section we show that this Leonard system has dual q -Krawtchouk type.

Theorem 23.1. *Let W denote an irreducible T -module. Let r, t, d denote the endpoint, dual endpoint and diameter of W , respectively. Let Φ denote the corresponding Leonard system on W from Lemma 14.8. Then Φ has dual q -Krawtchouk type. Let $h, h^*, \kappa, \kappa^*, v$ denote the parameters corresponding to Φ from Definition 11.1. Let ζ, ξ denote the scalars from Lemma 16.5. Then*

$$h = \frac{1 - q^{2e}}{q^2 - 1}, \quad h^* = \zeta, \quad \kappa = \frac{q^{2e+2D-2t-d}}{q^2 - 1}, \quad \kappa^* = \xi q^{-2r-d}, \quad v = -\frac{q^{2t+d}}{q^2 - 1}. \quad (78)$$

Proof: By Lemma 14.8 the Leonard system Φ and the T -module W have the same intersection matrix and dual intersection matrix. We show that Φ has dual q -Krawtchouk type by verifying that its parameter array satisfies (18)–(21). By Lemma 16.4 the eigenvalue sequence $\{\theta_{t+i}\}_{i=0}^d$ has the form (18) with scalars h, κ, v given in (78). By Lemma 16.5 the dual eigenvalue sequence $\{\theta_{r+i}^*\}_{i=0}^d$ has the form (19) with scalars h^*, κ^* given in (78). By Lemma 9.5 we have $\varphi_1 = (\theta_r^* - \theta_{r+1}^*)(a_0(W) - \theta_t)$. On the right-hand side evaluate the eigenvalue

using Lemma 16.4, evaluate the dual eigenvalues using Lemma 16.5, evaluate $a_0(W)$ using Lemma 17.2, and simplify the result to get

$$\varphi_1 = -\xi(q^{2d} - 1)q^{2(e+D-d-t-r-1)}. \quad (79)$$

By (PA4) we have $\phi_1 = \varphi_1 + (\theta_{r+1}^* - \theta_r^*)(\theta_{t+d} - \theta_t)$. On the right-hand side evaluate the eigenvalues using Lemma 16.4, evaluate the dual eigenvalues using Lemma 16.5, evaluate φ_1 using (79), and simplify the result to get

$$\phi_1 = \xi(q^{2d} - 1)q^{2(t-r-1)}. \quad (80)$$

On the right-hand side of (PA3) evaluate the eigenvalues using Lemma 16.4, evaluate the dual eigenvalues using Lemma 16.5, evaluate ϕ_1 using (80), and simplify the result to get

$$\varphi_i = \frac{\xi}{q^2 - 1}(q^{2(d+1)} - q^2)(q^{-2i} - 1)q^{2(e+D-d-t-r-i)} \quad (1 \leq i \leq d). \quad (81)$$

On the right-hand side of (PA4) evaluate the eigenvalues using Lemma 16.4, evaluate the dual eigenvalues using Lemma 16.5, evaluate φ_1 using (79), and simplify the result to get

$$\phi_i = \frac{\xi}{q^2 - 1}(q^{2(d+1)} - q^2)(q^{-2i} - 1)q^{2(e+D-d-t-r-i)} \quad (1 \leq i \leq d). \quad (82)$$

Using (78) it is routine to verify that (81), (82) satisfy (20), (21), respectively. Therefore the Leonard system Φ has dual q -Krawtchouk type. \square

24 Two $U_q(\mathfrak{sl}_2)$ -module structures on the standard module V

We continue to discuss the dual polar graph Γ from Section 16. In this section we display two $U_q(\mathfrak{sl}_2)$ -module structures on the standard module V . Then we show how the two $U_q(\mathfrak{sl}_2)$ -module structures are related.

Lemma 24.1. *Let W denote an irreducible T -module. Let r, t, d denote the endpoint, dual endpoint and diameter of W , respectively. Let $h, h^*, \kappa, \kappa^*, v$ denote the corresponding parameters from (78). Then there exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on W such that on W ,*

$$A = h1 + \kappa x + v y, \quad (83)$$

$$A^* = h^*1 + \kappa^* z. \quad (84)$$

Moreover the $U_q(\mathfrak{sl}_2)$ -module W is isomorphic to $L(d, 1)$.

Proof: Combine Theorem 23.1 and Theorem 13.19 taking $\epsilon = 1$. \square

Lemma 24.2. *Let W denote an irreducible T -module. Let r, t, d denote the endpoint, dual endpoint and diameter of W , respectively. Let $h, h^*, \kappa, \kappa^*, v$ denote the corresponding parameters from (78). Then there exists a unique $U_q(\mathfrak{sl}_2)$ -module structure on W such that on W ,*

$$\begin{aligned} A &= h1 + \kappa y + vx, \\ A^* &= h^*1 + \kappa^* z. \end{aligned}$$

Moreover the $U_q(\mathfrak{sl}_2)$ -module W is isomorphic to $L(d, 1)$.

Proof: Combine Theorem 23.1 and Theorem 13.20 taking $\epsilon = 1$. \square

Theorem 24.3. *There exists a $U_q(\mathfrak{sl}_2)$ -module structure on V such that on V ,*

$$\begin{aligned} A &= h1 + \kappa \Upsilon^{-1} \Psi x + v \Upsilon \Psi^{-1} y, \\ A^* &= h^*1 + \kappa^* \Upsilon^{-1} \Psi^{-1} z, \end{aligned} \tag{85}$$

$$\tag{86}$$

where

$$h = \frac{1 - q^{2e}}{q^2 - 1}, \quad h^* = \zeta, \quad \kappa = \frac{q^{2e+D}}{q^2 - 1}, \quad \kappa^* = \xi q^{-D}, \quad v = -\frac{q^D}{q^2 - 1}. \tag{87}$$

Proof: By construction V is a direct sum of irreducible T -modules. Let W denote an irreducible T -module in the sum. It suffices to show that (85), (86) hold on W . Let μ, ν denote the displacement of W of the first kind and second kind, respectively. Consider the $U_q(\mathfrak{sl}_2)$ -module structure on W from Lemma 24.1. Writing (83), (84) in terms of (87) and μ, ν we get

$$A = h1 + \kappa q^{\nu-\mu} x + v q^{\mu-\nu} y, \tag{88}$$

$$A^* = h^*1 + \kappa^* q^{-\mu-\nu} z, \tag{89}$$

where $h, h^*, \kappa, \kappa^*, v$ are from (87). By Lemma 22.5 we have $W \subseteq \sigma_\mu V$. By this and Lemma 22.8 the matrix Υ acts on W as $q^\mu I$. By Lemma 22.12 we have $W \subseteq \psi_\nu V$. By this and Lemma 22.15 the matrix Ψ acts on W as $q^\nu I$. By these comments and (88), (89) we find that (85), (86) hold on W . Therefore (85), (86) hold on V . \square

Theorem 24.4. *There exists a $U_q(\mathfrak{sl}_2)$ -module structure on V such that on V ,*

$$\begin{aligned} A &= h1 + \kappa \Upsilon^{-1} \Psi y + v \Upsilon \Psi^{-1} x, \\ A^* &= h^*1 + \kappa^* \Upsilon^{-1} \Psi^{-1} z, \end{aligned}$$

where the scalars $h, h^*, \kappa, \kappa^*, v$ are from (87).

Proof: Similar to the proof of Theorem 24.3 but use Lemma 24.2 instead of Lemma 24.1. \square

Lemma 24.5. For the $U_q(\mathfrak{sl}_2)$ -module structure on V from Theorem 24.3, the actions of x, y, z on V are given by

$$x = \frac{\Upsilon\Psi^{-1}(qB - q^{-1}B^*BB^{*-1})}{\kappa q^{-1}(q^2 - q^{-2})} + \frac{q^{-1}\kappa^*\Upsilon^{-1}\Psi^{-1}B^{*-1}}{q + q^{-1}} - \frac{q\kappa^*v\Upsilon\Psi^{-3}B^{*-1}}{\kappa(q + q^{-1})} \quad (90)$$

$$= \frac{\Upsilon\Psi^{-1}(qB^{*-1}BB^* - q^{-1}B)}{\kappa q(q^2 - q^{-2})} + \frac{q\kappa^*\Upsilon^{-1}\Psi^{-1}B^{*-1}}{q + q^{-1}} - \frac{q^{-1}\kappa^*v\Upsilon\Psi^{-3}B^{*-1}}{\kappa(q + q^{-1})},$$

$$y = \frac{\Upsilon^{-1}\Psi(qB - q^{-1}B^{*-1}BB^*)}{vq^{-1}(q^2 - q^{-2})} + \frac{q^{-1}\kappa^*\Upsilon^{-1}\Psi^{-1}B^{*-1}}{q + q^{-1}} - \frac{q\kappa^*\kappa\Upsilon^{-3}\Psi B^{*-1}}{v(q + q^{-1})} \quad (91)$$

$$= \frac{\Upsilon^{-1}\Psi(qB^*BB^{*-1} - q^{-1}B)}{vq(q^2 - q^{-2})} + \frac{q\kappa^*\Upsilon^{-1}\Psi^{-1}B^{*-1}}{q + q^{-1}} - \frac{q^{-1}\kappa^*\kappa\Upsilon^{-3}\Psi B^{*-1}}{v(q + q^{-1})},$$

$$z = \kappa^{*-1}\Upsilon\Psi B^*, \quad (92)$$

where $B = A - hI$, $B^* = A^* - h^*I$ and the scalars $h, h^*, \kappa, \kappa^*, v$ are from (87).

Proof: Similar to the proof of Lemma 13.14 but use Theorem 24.3 instead of Lemma 13.11. \square

Lemma 24.6. For the $U_q(\mathfrak{sl}_2)$ -module structure on V from Theorem 24.4, the actions of x, y, z on V are given by

$$x = \frac{\Upsilon^{-1}\Psi(qB - q^{-1}B^*BB^{*-1})}{vq^{-1}(q^2 - q^{-2})} + \frac{q^{-1}\kappa^*\Upsilon^{-1}\Psi^{-1}B^{*-1}}{q + q^{-1}} - \frac{q\kappa^*\kappa\Upsilon^{-3}\Psi B^{*-1}}{v(q + q^{-1})} \quad (93)$$

$$= \frac{\Upsilon^{-1}\Psi(qB^{*-1}BB^* - q^{-1}B)}{vq(q^2 - q^{-2})} + \frac{q\kappa^*\Upsilon^{-1}\Psi^{-1}B^{*-1}}{q + q^{-1}} - \frac{q^{-1}\kappa^*\kappa\Upsilon^{-3}\Psi B^{*-1}}{v(q + q^{-1})},$$

$$y = \frac{\Upsilon\Psi^{-1}(qB - q^{-1}B^{*-1}BB^*)}{\kappa q^{-1}(q^2 - q^{-2})} + \frac{q^{-1}\kappa^*\Upsilon^{-1}\Psi^{-1}B^{*-1}}{q + q^{-1}} - \frac{q\kappa^*v\Upsilon\Psi^{-3}B^{*-1}}{\kappa(q + q^{-1})} \quad (94)$$

$$= \frac{\Upsilon\Psi^{-1}(qB^*BB^{*-1} - q^{-1}B)}{\kappa q(q^2 - q^{-2})} + \frac{q\kappa^*\Upsilon^{-1}\Psi^{-1}B^{*-1}}{q + q^{-1}} - \frac{q^{-1}\kappa^*v\Upsilon\Psi^{-3}B^{*-1}}{\kappa(q + q^{-1})},$$

$$z = \kappa^{*-1}\Upsilon\Psi B^*, \quad (95)$$

where $B = A - hI$, $B^* = A^* - h^*I$ and the scalars $h, h^*, \kappa, \kappa^*, v$ are from (87).

Proof: Similar to the proof of Lemma 13.14 but use Theorem 24.4 instead of Lemma 13.11. \square

We draw several corollaries from Lemma 24.5 and Lemma 24.6.

Corollary 24.7. The $U_q(\mathfrak{sl}_2)$ -module structure on V from Theorem 24.3 is unique.

Proof: By Lemma 24.5 and since $x, y, z^{\pm 1}$ generate $U_q(\mathfrak{sl}_2)$. \square

Corollary 24.8. The $U_q(\mathfrak{sl}_2)$ -module structure on V from Theorem 24.4 is unique.

Proof: By Lemma 24.6 and since $x, y, z^{\pm 1}$ generate $U_q(\mathfrak{sl}_2)$. \square

Corollary 24.9. *For the \mathbb{C} -algebra homomorphism $U_q(\mathfrak{sl}_2) \rightarrow \text{End}(V)$ induced by the $U_q(\mathfrak{sl}_2)$ -module structure on V from Theorem 24.3, the image is contained in T .*

Proof: By Lemma 22.23 and Lemma 24.5. □

Corollary 24.10. *For the \mathbb{C} -algebra homomorphism $U_q(\mathfrak{sl}_2) \rightarrow \text{End}(V)$ induced by the $U_q(\mathfrak{sl}_2)$ -module structure on V from Theorem 24.4, the image is contained in T .*

Proof: By Lemma 22.23 and Lemma 24.6. □

In Theorem 24.3 and Theorem 24.4 we gave $U_q(\mathfrak{sl}_2)$ -actions on the standard module V . We now describe how these actions look on each irreducible T -module.

Lemma 24.11. *Let W denote an irreducible T -module. Then W is invariant under the $U_q(\mathfrak{sl}_2)$ -action from Theorem 24.3. Moreover the action of $U_q(\mathfrak{sl}_2)$ on W coincides with the $U_q(\mathfrak{sl}_2)$ -action from Lemma 24.1.*

Proof: The first assertion follows from Corollary 24.9. We now verify the second assertion. By Lemma 24.1 it suffices to show that (83), (84) hold on W . Let μ, ν denote the displacement of W of the first kind and second kind, respectively. By Lemma 22.5 we have $W \subseteq \sigma_\mu V$. By this and Lemma 22.8 the matrix Υ acts on W as $q^\mu I$. By Lemma 22.12 we have $W \subseteq \psi_\nu V$. By this and Lemma 22.15 the matrix Ψ acts on W as $q^\nu I$. By applying these comments, Definition 22.1 and (87) to (85), (86) we find that (83), (84) hold on W . □

Lemma 24.12. *Let W denote an irreducible T -module. Then W is invariant under the $U_q(\mathfrak{sl}_2)$ -action from Theorem 24.4. Moreover the action of $U_q(\mathfrak{sl}_2)$ on W coincides with the $U_q(\mathfrak{sl}_2)$ -action from Lemma 24.2.*

Proof: Similar to the proof of Lemma 24.11 but use Lemma 24.2 and Corollary 24.10 instead of Lemma 24.1 and Corollary 24.9. □

We finish this section with a comment describing how the two $U_q(\mathfrak{sl}_2)$ -module structures on V from Theorem 24.3 and Theorem 24.4 are related.

Theorem 24.13. *Consider the table below. In the first column the three displayed elements each induces an element in $\text{End}(V)$ using the $U_q(\mathfrak{sl}_2)$ -module structure from Theorem 24.4. In the second column the three displayed elements each induces an element in $\text{End}(V)$ using the $U_q(\mathfrak{sl}_2)$ -module structure from Theorem 24.3. For each row the two elements induce the same element of $\text{End}(V)$.*

$U_q(\mathfrak{sl}_2)$ -module structure from Theorem 24.4	$U_q(\mathfrak{sl}_2)$ -module structure from Theorem 24.3
z	z
x	$-q^{2e}\Upsilon^{-2}\Psi^2x + (1 + q^{2e}\Upsilon^{-2}\Psi^2)z^{-1}$
y	$-q^{-2e}\Upsilon^2\Psi^{-2}y + (1 + q^{-2e}\Upsilon^2\Psi^{-2})z^{-1}$

Proof: The first row is immediate from (92), (95). To prove the second row evaluate x on the left using (93), and evaluate x, z on the right using (90), (92). To prove the last row evaluate y on the left using (94), and evaluate y, z on the right using (91), (92). \square

25 Two homomorphisms $U_q(\mathfrak{sl}_2) \rightarrow T$

We continue to discuss the dual polar graph Γ from Section 16. In Theorem 24.3 and Theorem 24.4 we displayed two $U_q(\mathfrak{sl}_2)$ -module structures on the standard module V . In this section we show how these two $U_q(\mathfrak{sl}_2)$ -module structures are related to T . By Corollary 24.9 the $U_q(\mathfrak{sl}_2)$ -module structure from Theorem 24.3 induces a \mathbb{C} -algebra homomorphism $U_q(\mathfrak{sl}_2) \rightarrow T$. By Corollary 24.10 the $U_q(\mathfrak{sl}_2)$ -module structure from Theorem 24.4 induces a \mathbb{C} -algebra homomorphism $U_q(\mathfrak{sl}_2) \rightarrow T$. For either of the two \mathbb{C} -algebra homomorphisms, let U denote the image. We show that T is generated by U together with the elements $\Upsilon^{\pm 1}, \Psi^{\pm 1}$ where Υ is from Definition 22.7 and Ψ is from Definition 22.14.

Recall below Lemma 12.13 we discussed the homogeneous components for a $U_q(\mathfrak{sl}_2)$ -module. We now consider the homogeneous components for the $U_q(\mathfrak{sl}_2)$ -module structure on V from either Theorem 24.3 or Theorem 24.4.

Lemma 25.1. *Consider the $U_q(\mathfrak{sl}_2)$ -module structure on V from either Theorem 24.3 or Theorem 24.4. Then*

$$V_{d,-1} = 0, \quad V_{d,1} = \rho_d V \quad (0 \leq d \leq D). \quad (96)$$

In other words, in the sum (76) the summands are the homogeneous components of the $U_q(\mathfrak{sl}_2)$ -module V .

Proof: First assume the $U_q(\mathfrak{sl}_2)$ -module structure on V is from Theorem 24.3. Recall from Lemma 22.19 that $\rho_d V$ is spanned by the irreducible T -modules of diameter d . By Lemma 24.11 and Lemma 24.1 each irreducible T -module of diameter d is a $U_q(\mathfrak{sl}_2)$ -module isomorphic to $L(d, 1)$. By these comments,

$$\rho_d V \subseteq V_{d,1}. \quad (97)$$

By summing (97) over $0 \leq d \leq D$ and comparing the result to (76) we have

$$V = \sum_{d=0}^D V_{d,1} \quad (\text{direct sum}). \quad (98)$$

By comparing (98) to (30) with $M = V$ we have (96). We are now done for the case in which the module structure is from Theorem 24.3. For the case in which the module structure is from Theorem 24.4, the argument is similar using Lemma 24.2 and Lemma 24.12 instead of Lemma 24.1 and Lemma 24.11. \square

In the beginning of this section we discussed two \mathbb{C} -algebra homomorphisms $U_q(\mathfrak{sl}_2) \rightarrow T$. We now discuss these homomorphisms.

Theorem 25.2. *For either of our two \mathbb{C} -algebra homomorphisms, let U denote the image. Then the algebra T is generated by U together with the elements $\Upsilon^{\pm 1}, \Psi^{\pm 1}$ where Υ is from Definition 22.7 and Ψ is from Definition 22.14.*

Proof: By Theorem 24.3, Theorem 24.4, and since A, A^* generate T . \square

We finish this section with a comment. Recall the Casimir element Δ of $U_q(\mathfrak{sl}_2)$ from Definition 12.10.

Lemma 25.3. *For the $U_q(\mathfrak{sl}_2)$ -module structure on V from either Theorem 24.3 or Theorem 24.4, the Casimir element Δ acts on V as the element Λ from Definition 22.21.*

Proof: By (76), for $0 \leq d \leq D$ it suffices to show that Δ, Λ agree on $\rho_d V$. Recall from Lemma 22.22 that the element Λ acts on $\rho_d V$ as (77). By Lemma 25.1 and Lemma 12.12, the element Δ acts on $\rho_d V$ as (77). Therefore they agree on $\rho_d V$. The result follows. \square

26 The matrices L, F, R, K

We continue to discuss the dual polar graph Γ from Section 16. In this section we define some nice matrices that generate T and find relations among them.

Definition 26.1. We define the matrices L, F, R in T by

$$L = \sum_{i=1}^D E_{i-1}^* A E_i^*, \quad (99)$$

$$F = \sum_{i=0}^D E_i^* A E_i^*, \quad (100)$$

$$R = \sum_{i=0}^{D-1} E_{i+1}^* A E_i^*. \quad (101)$$

We call L (resp. F) (resp. R) the *lowering matrix* (resp. *flattening matrix*) (resp. *raising matrix*).

Observe that $L^t = R$ and $F^t = F$. Moreover $A = L + F + R$.

Lemma 26.2. *The following (i)–(iii) hold.*

- (i) $LE_i^* = E_{i-1}^* LE_i^* = E_{i-1}^* L = E_{i-1}^* A E_i^* \quad (1 \leq i \leq D), \quad LE_0^* = 0.$
- (ii) $FE_i^* = E_i^* FE_i^* = E_i^* F = E_i^* A E_i^* \quad (0 \leq i \leq D).$
- (iii) $RE_i^* = E_{i+1}^* RE_i^* = E_{i+1}^* R = E_{i+1}^* A E_i^* \quad (0 \leq i \leq D-1), \quad RE_D^* = 0.$

Proof: Routine verification using Definition 26.1 and $E_i^* E_j^* = \delta_{ij} E_i^* \quad (0 \leq i, j \leq D).$ \square

Proposition 26.3. *The following (i), (ii) hold.*

$$(i) \quad LF - q^2 FL = (q^{2e} - 1)L.$$

$$(ii) \quad FR - q^2 RF = (q^{2e} - 1)R.$$

Proof: (i) Let y, z denote vertices in X . Let $i = \partial(x, z)$. For the equation in (i) we show that the (y, z) -entries of both sides are equal. By construction

$$L_{yz} = \begin{cases} 1 & \text{if } \partial(y, z) = 1 \text{ and } \partial(x, y) = i - 1; \\ 0 & \text{otherwise.} \end{cases}$$

By Definition 26.1 and Lemma 20.4,

$$(LF)_{yz} = \begin{cases} 0 & \text{if } \partial(x, y) \neq i - 1; \\ |\Gamma_i(x) \cap \Gamma(y) \cap \Gamma(z)| & \text{if } \partial(x, y) = i - 1, \end{cases}$$

$$(FL)_{yz} = \begin{cases} 0 & \text{if } \partial(x, y) \neq i - 1; \\ |\Gamma_{i-1}(x) \cap \Gamma(y) \cap \Gamma(z)| & \text{if } \partial(x, y) = i - 1. \end{cases}$$

We split our argument into cases.

Case $y = z$, or $\partial(y, z) > 2$, or $\partial(x, y) \neq i - 1$: The (y, z) -entries of LF, FL and L are zero. Therefore the (y, z) -entry of each side of the equation in (i) is zero.

Case $\partial(y, z) = 2$ and $\partial(x, y) = i - 1$: By Lemma 18.3 the (y, z) -entry of LF is q^2 and the (y, z) -entry of FL is 1. The (y, z) -entry of L is zero. Therefore the (y, z) -entry of each side of the equation in (i) is zero.

Case $\partial(y, z) = 1$ and $\partial(x, y) = i - 1$: By Lemma 15.5 and Corollary 16.2 the (y, z) -entry of LF is $q^{2e} - 1$ and the (y, z) -entry of FL is 0. The (y, z) -entry of L is 1. Therefore the (y, z) -entry of each side of the equation in (i) is $q^{2e} - 1$.

(ii) Take the transpose of each term in (i). □

Proposition 26.4. *The following (i), (ii) hold.*

$$(i) \quad \frac{q^4}{q^2 + 1} RL^2 - LRL + \frac{q^{-2}}{q^2 + 1} L^2 R = -q^{2e+2D-2} L.$$

$$(ii) \quad \frac{q^4}{q^2 + 1} R^2 L - RLR + \frac{q^{-2}}{q^2 + 1} LR^2 = -q^{2e+2D-2} R.$$

Proof: (i) Let

$$S = \frac{q^4}{q^2 + 1} RL^2 - LRL + \frac{q^{-2}}{q^2 + 1} L^2 R + q^{2e+2D-2} L.$$

We show $S = 0$. Since $I = \sum_{i=0}^D E_i^*$,

$$S = \left(\sum_{i=0}^D E_i^* \right) S \left(\sum_{j=0}^D E_j^* \right) = \sum_{i=0}^D \sum_{j=0}^D E_i^* S E_j^*.$$

By Lemma 26.2 and since $E_l^* E_m^* = \delta_{lm} E_l^*$ for $0 \leq l, m \leq D$, we have $E_i^* S E_j^* = 0$ if $i \neq j - 1$. Therefore it suffices to show $E_{j-1}^* S E_j^* = 0$ for $1 \leq j \leq D$. Let j be given. By Lemma 16.5,

$$\theta_l^* - \theta_{l-1}^* = q^2(\theta_{l+1}^* - \theta_l^*) \quad (1 \leq l \leq D-1). \quad (102)$$

Expanding (61) we get

$$\begin{aligned} 0 &= A^3 A^* - (\beta + 1) A^2 A^* A + (\beta + 1) A A^* A^2 - A^* A^3 \\ &\quad + \gamma(A^* A^2 - A^2 A^*) + \varrho(A^* A - A A^*). \end{aligned}$$

In the above equation multiply each term on the left by E_{j-1}^* and on the right by E_j^* . Simplify the result using $A = L + F + R$, and $A^* E_l^* = \theta_l^* E_l^*$ ($0 \leq l \leq D$) along with Lemma 16.7(i), Lemma 26.2 and (102). This yields

$$\begin{aligned} 0 &= (q + q^{-1})^2 E_{j-1}^* \left(-\frac{q^4}{q^2 + 1} R L^2 + L R L - \frac{q^{-2}}{q^2 + 1} L^2 R \right) E_j^* \\ &\quad + E_{j-1}^* \left(L F^2 - \beta F L F + F^2 L - \gamma(F L + L F) - \varrho L \right) E_j^*. \end{aligned} \quad (103)$$

Using Lemma 16.7 and Proposition 26.3(i) one checks

$$L F^2 - \beta F L F + F^2 L - \gamma(F L + L F) - \varrho L = -q^{2D+2e-2}(q + q^{-1})^2 L. \quad (104)$$

Simplifying (103) using (104) and $q + q^{-1} \neq 0$, we have $E_{j-1}^* S E_j^* = 0$. We have now shown that $S = 0$. The result follows.

(ii) Take the transpose of each term in (i). □

Definition 26.5. We define the matrix $K \in T$ by

$$K = \sum_{i=0}^D q^{-2i} E_i^*. \quad (105)$$

Observe that K is diagonal. Moreover $K = \xi^{-1}(A^* - \zeta I)$ where ζ, ξ are from Lemma 16.5.

The next two lemmas follow from Definition 26.5.

Lemma 26.6. *The matrix K is invertible and $K^{-1} = \sum_{i=0}^D q^{2i} E_i^*$.*

Lemma 26.7. *For $0 \leq i \leq D$ we have $K E_i^* = E_i^* K = q^{-2i} E_i^*$.*

Lemma 26.8. *The algebra T is generated by L, F, R, K .*

Proof: Recall that T is generated by A, A^* . The result follows from this, the comments after Definition 26.1, and the comments after Definition 26.5. □

Proposition 26.9. *The following (i)–(iii) hold.*

$$(i) \quad KL = q^2LK.$$

$$(ii) \quad KF = FK.$$

$$(iii) \quad KR = q^{-2}RK.$$

Proof: (i) By (105) we have $KL = \sum_{i=0}^D q^{-2i} E_i^* L$ and $q^2LK = \sum_{i=0}^d q^{2-2i} LE_i^*$. Comparing these equations using Lemma 26.2(i) we find $KL = q^2LK$.

(ii) Similar to the proof of (i).

(iii) Take the transpose of each term in (i). □

We mention a consequence of the relations from Proposition 26.3, Proposition 26.4 and Proposition 26.9.

Lemma 26.10. *The matrices LR, RL, F, K mutually commute.*

Proof: By Proposition 26.9 the matrix K commutes with each of LR, RL, F . By Proposition 26.3 the matrix F commutes with each of LR, RL . It remains to show that LR, RL commute. In Proposition 26.4 multiply each side of equation (i) on the right by R and multiply each side of equation (ii) on the left by L . Taking the difference between the resulting equations and simplifying we get $LR^2L = RL^2R$. Therefore LR, RL commute. The result follows. □

The matrices L, F, R can be expressed in terms of A, K, K^{-1} as follows.

Lemma 26.11. *The following (i)–(iii) hold.*

$$(i) \quad L = \frac{q^{-1}K^{-1}AK + qKAK^{-1} - (q + q^{-1})A}{(q - q^{-1})^2(q + q^{-1})}.$$

$$(ii) \quad F = \frac{(q^2 + q^{-2})A - K^{-1}AK - KAK^{-1}}{(q - q^{-1})^2}.$$

$$(iii) \quad R = \frac{qK^{-1}AK + q^{-1}KAK^{-1} - (q + q^{-1})A}{(q - q^{-1})^2(q + q^{-1})}.$$

Proof: In each equation eliminate A using $A = L + F + R$ and simplify the result using Proposition 26.9. □

27 The central elements Ω, G, G^* in terms of L, F, R, K

We continue to discuss the dual polar graph Γ from Section 16. Recall the central elements Ω, G, G^* of T from Lemma 14.11. Recall the elements L, F, R, K of T from Section 26. In this section we display each of Ω, G, G^* in terms of L, F, R, K .

Proposition 27.1. *Let ξ, ζ denote the scalars from Lemma 16.5, and let γ, ϱ denote the scalars from Lemma 16.7. Then Ω, G, G^* can be expressed in terms of L, F, R, K as follows.*

- (i) $\Omega = -\frac{\xi(q^2 - 1)^2}{q^2}KF - \frac{\xi(q^2 - 1)(q^{2e} - 1)}{q^2}K - 2\gamma\zeta I.$
- (ii) $G = \xi(1 - q^4)KRL + \xi(1 - q^{-4})KLR + \xi(q^{-2} - 1)(q^{2e} - 1)KF - \varrho\xi K - \varrho\zeta I.$
- (iii) $G^* = \frac{\zeta\xi(q^2 - 1)^2}{q^2}KF + \frac{\zeta\xi(q^2 - 1)(q^{2e} - 1)}{q^2}K + \gamma\zeta^2 I.$

Proof: (i) Combine Lemma 20.2, Definition 26.1 and Definition 26.5.

(ii) Combine Lemma 21.1, Definition 26.1 and Definition 26.5.

(iii) Combine Lemma 20.1 and (i). □

28 The central elements C_0, C_1, C_2

We continue to discuss the dual polar graph Γ from Section 16. In this section we define three matrices C_0, C_1, C_2 in T which involve L, F, R, K from Section 26. We show that C_0, C_1, C_2 are in $Z(T)$. Then we display the actions of C_0, C_1, C_2 on each irreducible T -module. Using this data we show that C_0, C_1, C_2 generate $Z(T)$.

Definition 28.1. We define the matrices C_0, C_1, C_2 in T as follows.

- (i) $C_0 = KF + \frac{q^{2e} - 1}{q^2 - 1}K.$
- (ii) $C_1 = \frac{q^2}{q^2 + 1}KRL - \frac{q^{-2}}{q^2 + 1}KLR + \frac{q^{2e+2D-2}}{q^2 - 1}K.$
- (iii) $C_2 = \frac{1}{q^2 + 1}K^2RL - \frac{q^{-2}}{q^2 + 1}K^2LR + \frac{q^{2e+2D-2}}{q^4 - 1}K^2.$

We have an observation.

Lemma 28.2. *The matrices C_0, C_1, C_2 are symmetric.*

Proof: By Lemma 26.10 and Definition 28.1 together with the fact that F, K are symmetric and $R^t = L$. □

Lemma 28.3. *Each of C_0, C_1, C_2 is in $Z(T)$.*

Proof: We first show that $C_0 \in Z(T)$. By Lemma 26.8 it suffices to show that C_0 commutes with each of L, F, R, K . By Lemma 26.10 the matrix C_0 commutes with F, K . Using Proposition 26.3(ii) and Proposition 26.9(iii) one checks that C_0 commutes with R . By this, Lemma 28.2 and since $R^t = L$, we have $C_0L - LC_0 = (RC_0 - C_0R)^t = 0$. Therefore C_0 commutes with L . We have now shown that C_0 commutes with each of L, F, R, K . Therefore $C_0 \in Z(T)$. By a similar argument using Proposition 26.4, Proposition 26.9 and Lemma 26.10 we find that C_1, C_2 are in $Z(T)$. □

Lemma 28.4. *Let W denote an irreducible T -module. Let r, t, d denote the endpoint, dual endpoint and diameter of W , respectively. Then the following (i)–(iii) hold.*

(i) C_0 acts on W as $\chi_0(r, t, d)I$ where

$$\chi_0(r, t, d) = \frac{q^{2e+2D-2d-2r-2t} - q^{2t-2r}}{q^2 - 1}. \quad (106)$$

(ii) C_1 acts on W as $\chi_1(r, t, d)I$ where

$$\chi_1(r, t, d) = \frac{q^{2e+2D-1-d-2r}(q^{d+1} + q^{-d-1})}{q^4 - 1}. \quad (107)$$

(iii) C_2 acts on W as $\chi_2(r, t, d)I$ where

$$\chi_2(r, t, d) = \frac{q^{2e+2D-2-2d-4r}}{q^4 - 1}. \quad (108)$$

Proof: Fix an integer i ($0 \leq i \leq d$). By Definition 28.1(i), (100), (105) and Definition 14.5, the element C_0 acts on E_{r+i}^*W as

$$q^{-2r-2i}a_i(W) + \frac{q^{-2r-2i}(q^{2e} - 1)}{q^2 - 1} \quad (109)$$

times I . Using Lemma 17.2 one checks that (109) equals $\chi_0(r, t, d)$. Therefore (i) holds. By Definition 28.1(ii), (99), (101), (105) and Definition 14.5, the element C_1 acts on E_{r+i}^*W as

$$\frac{q^{2-2r-2i}}{q^2 + 1}c_i(W)b_{i-1}(W) - \frac{q^{-2-2r-2i}}{q^2 + 1}b_i(W)c_{i+1}(W) + \frac{q^{2e+2D-2-2r-2i}}{q^2 - 1} \quad (110)$$

times I . Using Lemma 17.2 one checks that (110) equals $\chi_1(r, t, d)$. Therefore (ii) holds. By Definition 28.1(iii), (99), (101), (105) and Definition 14.5, the element C_2 acts on E_{r+i}^*W as

$$\frac{q^{-4r-4i}}{q^2 + 1}c_i(W)b_{i-1}(W) - \frac{q^{-2-4r-4i}}{q^2 + 1}b_i(W)c_{i+1}(W) + \frac{q^{2e+2D-2-4r-4i}}{q^4 - 1} \quad (111)$$

times I . Using Lemma 17.2 one checks that (111) equals $\chi_2(r, t, d)$. Therefore (iii) holds. \square

Theorem 28.5. *The algebra $Z(T)$ is generated by C_0, C_1, C_2 .*

Proof: Let Z' denote the subalgebra of T generated by C_0, C_1, C_2 . We show $Z' = Z(T)$. By Lemma 28.3, the algebra Z' is contained in $Z(T)$. We now show the reverse inclusion. To do this, by Lemma 19.1 it suffices to show $E_\lambda \in Z'$ for all $\lambda \in \text{Feas}$. Let (r, t, d) and (r', t', d') denote distinct elements of Feas . We claim that there exists an integer $i \in \{0, 1, 2\}$ such that $\chi_i(r, t, d) \neq \chi_i(r', t', d')$. Suppose this is not the case. By (106)–(108),

$$q^{2e+2D-2d-2r-2t} - q^{2t-2r} = q^{2e+2D-2d'-2r'-2t'} - q^{2t'-2r'}, \quad (112)$$

$$q^{-d-2r}(q^{d+1} + q^{-d-1}) = q^{-d'-2r'}(q^{d'+1} + q^{-d'-1}), \quad (113)$$

$$q^{-2d-4r} = q^{-2d'-4r'}. \quad (114)$$

We show $d = d'$. By (114) we have $q^{-d-2r} = q^{-d'-2r'}$. By this and (113) we have $q^{d+1} + q^{-d-1} = q^{d'+1} + q^{-d'-1}$. Simplifying this we get $(q^{d+d'+2} - 1)(q^{d+1} - q^{d'+1}) = 0$. Since $d+d'+2 \neq 0$ and q is not a root of unity, we have $q^{d+1} = q^{d'+1}$ so $d = d'$. Next we show $r = r'$. This is immediate from (114) and the fact that $d = d'$. Now we show $t = t'$. Evaluate (112) using $r = r'$ and $d = d'$, and simplify the result we get $(q^{2e+2D-2d} + q^{2t+2t'})(q^{2t'} - q^{2t}) = 0$. But $q^{2e+2D-2d} + q^{2t+2t'} \neq 0$ since q is not a root of unity. Therefore $q^{2t'} = q^{2t}$ and thus $t = t'$. We have shown $(r, t, d) = (r', t', d')$ for a contradiction. Therefore the claim holds. By the claim and Lemma 28.4 we have $E_\lambda \in Z'$ for all $\lambda \in \text{Feas}$. The result follows. \square

We finish this section with a comment.

Lemma 28.6. *The following (i), (ii) hold.*

$$\begin{aligned} \text{(i)} \quad RL &= \frac{q^2 + 1}{q^2 - 1} K^{-1} C_1 - \frac{q^2 + 1}{q^2 - 1} K^{-2} C_2 - \frac{q^{2e+2D}}{(q^2 - 1)^2} I. \\ \text{(ii)} \quad LR &= \frac{q^2(q^2 + 1)}{q^2 - 1} K^{-1} C_1 - \frac{q^4(q^2 + 1)}{q^2 - 1} K^{-2} C_2 - \frac{q^{2e+2D}}{(q^2 - 1)^2} I. \end{aligned}$$

Proof: Solve for RL and LR using Definition 28.1(ii), (iii). \square

29 How C_0, C_1, C_2 relate to Ω, G, G^* and Υ, Ψ, Λ

We continue to discuss the dual polar graph Γ from Section 16. So far we obtained a number of central elements of T . We have Ω, G, G^* from Lemma 14.11, Υ, Ψ, Λ from Section 22, and C_0, C_1, C_2 from Section 28. In Proposition 22.24 we expressed Ω, G, G^* in terms of Υ, Ψ, Λ . In this section we express Ω, G, G^* in terms of C_0, C_1, C_2 and express C_0, C_1, C_2 in terms of Υ, Ψ, Λ .

Proposition 29.1. *Let ζ, ξ denote the scalars from Lemma 16.5 and let γ, ϱ denote the scalars from Lemma 16.7. Then the following (i)–(iii) hold.*

$$\begin{aligned} \text{(i)} \quad \Omega &= -\xi q^{-2}(q^2 - 1)^2 C_0 - 2\gamma \zeta I. \\ \text{(ii)} \quad G &= \xi(q^{-2} - 1)(q^{2e} - 1)C_0 + \xi(q^{-2} - 1)(q^2 + 1)^2 C_1 - \varrho \zeta I. \\ \text{(iii)} \quad G^* &= \zeta \xi q^{-2}(q^2 - 1)^2 C_0 + \gamma \zeta^2 I. \end{aligned}$$

Proof: (i) Evaluate Ω using Proposition 27.1(i). Evaluate C_0 using Definition 28.1.

(ii) Evaluate G using Proposition 27.1(ii). Evaluate C_0, C_1 using Definition 28.1.

(iii) Combine Lemma 20.1 and (i). \square

Proposition 29.2. *The matrices C_0, C_1, C_2 can be expressed in terms of Υ, Ψ, Λ as follows.*

$$\begin{aligned} \text{(i)} \quad C_0 &= (q^2 - 1)^{-1}(q^{2e}\Upsilon^{-2} - \Psi^{-2}). \\ \text{(ii)} \quad C_1 &= q^{2e+D-3}(q^2 - 1)(q^2 + 1)^{-1}\Upsilon^{-1}\Psi^{-1}\Lambda. \end{aligned}$$

$$(iii) \ C_2 = q^{2e-2}(q^4 - 1)^{-1}\Upsilon^{-2}\Psi^{-2}.$$

Proof: (i) By construction V is a direct sum of irreducible T -modules. Let W denote an irreducible T -module in the sum. It suffices to show that for the equation in (i) the two sides agree on W . Let r, t, d denote the endpoint, dual endpoint, and diameter of W , respectively. By Lemma 28.4, the element C_0 acts on W as $\chi_0(r, t, d)I$. By Lemma 22.5 we have $W \subseteq \sigma_{r+t+d-D}V$. By this and Lemma 22.8 the matrix Υ acts on W as $q^{r+t+d-D}1$. By Lemma 22.12 we have $W \subseteq \psi_{r-t}V$. By this and Lemma 22.15 the matrix Ψ acts on W as $q^{r-t}1$. By these comments and (106), the two sides agree on W . The result follows
(ii), (iii) Similar to the proof of (i). \square

30 L, F, R, K and $U_q(\mathfrak{sl}_2)$

We continue to discuss the dual polar graph Γ from Section 16. In this section we display some relationships between the $U_q(\mathfrak{sl}_2)$ -module structures from Theorem 24.3 and Theorem 24.4 and the matrices L, F, R, K from Section 26.

Recall the central elements Υ, Ψ of T from Section 22.

Lemma 30.1. *The actions of x, y, z on the $U_q(\mathfrak{sl}_2)$ -module V from Theorem 24.3 are given by*

$$\begin{aligned} x &= q^{-D}\Upsilon^{-1}\Psi^{-1}K^{-1} + q^{-2e-D}(q^2 - 1)\Upsilon\Psi^{-1}R, \\ y &= q^{-D}\Upsilon^{-1}\Psi^{-1}K^{-1} - q^{-D}(q^2 - 1)\Upsilon^{-1}\Psi L, \\ z &= q^D\Upsilon\Psi K. \end{aligned}$$

Proof: Let h denote the scalar from (87) and let ξ denote the scalar from Lemma 16.5. By Lemma 28.1 we have $F - hI = K^{-1}C_0$. By this and Proposition 29.2 we have

$$F - hI = (q^2 - 1)^{-1}(q^{2e}\Upsilon^{-2} - \Psi^{-2})K^{-1}. \quad (115)$$

In each of (90)–(92) evaluate B, B^* using $B = L + F + R - hI$ and $B^* = \xi K$. Simplify the result using Proposition 26.9 and (115). \square

Lemma 30.2. *The actions of x, y, z on the $U_q(\mathfrak{sl}_2)$ -module V from Theorem 24.4 are given by*

$$\begin{aligned} x &= q^{-D}\Upsilon^{-1}\Psi^{-1}K^{-1} - q^{-D}(q^2 - 1)\Upsilon^{-1}\Psi R, \\ y &= q^{-D}\Upsilon^{-1}\Psi^{-1}K^{-1} + q^{-2e-D}(q^2 - 1)\Upsilon\Psi^{-1}L, \\ z &= q^D\Upsilon\Psi K. \end{aligned}$$

Proof: Similar to the proof of Lemma 30.1 but use (93)–(95) instead of (90)–(92). \square

We now give reformulations of Lemma 30.1 and Lemma 30.2 in terms of the generators k, e, f for $U_q(\mathfrak{sl}_2)$.

Lemma 30.3. *The actions of k, e, f on the $U_q(\mathfrak{sl}_2)$ -module V from Theorem 24.3 are given by*

$$\begin{aligned} k &= q^D \Upsilon \Psi K, \\ e &= \Psi^2 K L, \\ f &= q^{1-2e-D} \Upsilon \Psi^{-1} R. \end{aligned}$$

Proof: Combine Lemma 12.2 and Lemma 30.1. □

Lemma 30.4. *The actions of k, e, f on the $U_q(\mathfrak{sl}_2)$ -module V from Theorem 24.4 are given by*

$$\begin{aligned} k &= q^D \Upsilon \Psi K, \\ e &= -q^{-2e} \Upsilon^2 K L, \\ f &= -q^{1-D} \Upsilon^{-1} \Psi R. \end{aligned}$$

Proof: Combine Lemma 12.2 and Lemma 30.2. □

We finish this section with a comment describing how the two actions of $U_q(\mathfrak{sl}_2)$ on V from Theorem 24.3 and Theorem 24.4 are related.

Theorem 30.5. *Consider the table below. In the first column the three displayed elements each induces an element in $\text{End}(V)$ using the $U_q(\mathfrak{sl}_2)$ -module structure from Theorem 24.4. In the second column the three displayed elements each induces an element in $\text{End}(V)$ using the $U_q(\mathfrak{sl}_2)$ -module structure from Theorem 24.3. For each row the two elements induce the same element of $\text{End}(V)$.*

$U_q(\mathfrak{sl}_2)$ -module structure from Theorem 24.4	$U_q(\mathfrak{sl}_2)$ -module structure from Theorem 24.3
k	k
e	$-q^{-2e} \Upsilon^2 \Psi^{-2} e$
f	$-q^{2e} \Upsilon^{-2} \Psi^2 f$

Proof: Compare the $U_q(\mathfrak{sl}_2)$ -actions from Lemma 30.3 and Lemma 30.4. □

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